Math 2280-002
Week 8-9 concepts and homework, due Wednesday March 6

Recall that all listed problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Wednesday quiz will be drawn from all of these concepts and from these or related problems.

3.6 practical resonance
3.6: 15

4.1: modeling coupled mass-spring systems or multi-component input-output systems with systems of differential equations;converting single differential equations or systems of differential equations into equivalent first order systems of differential equations by introducing functions for the intermediate derivatives; comparing solutions to these equivalent systems.
4.1: 1, 3, **2**, 5, **8**, 17, 27, **30**, **31**, **32**.

5.1: recognizing homogeneous and non-homogeneous linear systems of first order differential equations; writing these systems in vector-matrix form; statement of existence and uniqueness for IVP's in first order systems of DE's and its consequences for the dimension of the solution space to the first order system, and for the general solution to the non-homogeneous system. Using the Wronskian to check for bases.
5.1: 11, **12**, **13**, 21, **22**, **31.**

5.2: the eigenvalue-eigenvector method for finding the solution space to homogeneous constant coefficient first order systems of differential equations: real and complex eigenvalues.
5.2: 3, 13, 29, 31, **34**, 36. *In* **34** *you may use technology to find or check the eigendata, as long as you're confident you could do the work by hand.*

w8.1) (Section 3.6) Consider the following forced oscillator equation with a relatively small damping coefficient of $c = .2\ \frac{kg}{s}$. As usual in section 3.6 we will consider sinusoidal forcing, but unlike in last week's homework, this time we will allow the forcing function to have variable angular frequencies ω. Precisely, consider the following differential equation

$$x'' + .2\,x' + 0.64\,x = 2\cos(\omega\,t)\ .$$

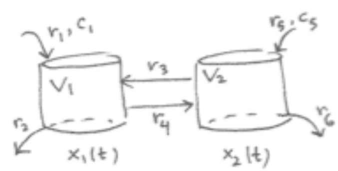
a) Use the formula (21) in section 3.6 of the text (which we also discussed in class on Tuesday February 26) and technology (Desmos is great) to create a plot of the amplitude C of the steady periodic solution $x_{sp}(t)$ as a function of the driving angular frequency ω.
b) Explain why the steady periodic amplitude peaks at a value near $\omega = 0.8$. Use calculus to find the exact value of ω which gives the maximum possible amplitude for the steady periodic solution, as the driving angular frequency varies.

w8.2) This is a continuation of **5.1.22**, **5.1.31**.
a) Use the eigenvalue-eigenvector ($e^{\lambda t}\underline{\boldsymbol{v}}$) method of section 5.2 to generate the basis $\{\underline{\boldsymbol{x}}_1(t), \underline{\boldsymbol{x}}_2(t)\}$ for the general solution that the text tells you in **5.1.22**.
b) Use pplane to draw the phase portrait for this first order system along with the parametric curve of the solution $[x(t), y(t)]^T$ to the initial value problem in **31**. Print out a screen shot of your work to hand in.

w8.3) (Postponed until next assignment) Use the eigenvalue-eigenvector method (with complex eigenvalues) to solve the first order system initial value problem which is equivalent to the second order differential equation IVP in the Monday March 4 notes. This is the reverse procedure from Monday, when we use the solutions from the equivalent second order DE IVP to deduce the solution to the first order system IVP. Of course, your answer here should be consistent with our work there.

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

w8.4) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t), x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i, $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



a) What equalities between the flow rates guarantee that the volumes V_1, V_2 remain constant?

b) Assuming the equalities in **a** hold, what first order system of differential equations governs the rates of change for $x_1(t), x_2(t)$?

c) Suppose $r_2 = r_4 = r_6 = 100,\ r_3 = r_5 = 200,\ r_1 = 0\ \dfrac{gal}{hour}$; $c_1 = 0,\ c_5 = 0.3\ \dfrac{lb}{gal}$; $V_1 = V_2 = 100\ gal$.
Verify that the constant volumes are consistent with the rate balancing required in **a**. Then show that the general system in **b** reduces to the following system of DEs for the given parameter values:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 60 \end{bmatrix}.$$

d) Solve the initial value problem for **c**, assuming there is initially no solute in either tank. Hint: Find the homogeneous solution; then find a particular solution which is a constant vector; and then use $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ to solve the IVP.

e) Re-solve the IVP in part **d** with the direct approach based on diagonalization of the matrix

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$$

in this system of DEs: let P be a matrix of eigenvector columns that diagonalizes A, i.e. $AP = PD$ where D is the corresponding diagonal matrix with eigenvalues along the diagonal. Make the change of functions

$$P\,\underline{\boldsymbol{u}}(t) = \underline{\boldsymbol{x}}(t)$$

for the IVP in part (c). Then directly solve for $\underline{\boldsymbol{u}}(t)$ using Chapter 1 methods, and recover $\underline{\boldsymbol{x}}(t)$. This is like a discussion we have in class on Friday March 1, except we carry it out there for a homogeneous system of DE's.

f) Check your answer to **d,e** with technology (e.g. Wolfram alpha) and hand in a copy of this verification.