Math 2280-002
Week 7-8 concepts and homework, due Wednesday February 27

Recall that all listed problems are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Wednesday quizzes will be drawn from all of these concepts and from these or related problems.

3.4: applications for unforced mechanical systems.
3.4: **5**, **6**, **23**

3.5: finding particular solutions using the method of undetermined coefficients; using the general solution $y = y_p + y_H$ to solve associated initial value problems.
3.5: 2, 3, 10, 21, 27, 29, **31**, 34, **43**, 52.

3.6: *solving forced oscillation problems; understanding beating and resonance in undamped problems, steady periodic and transient solutions in damped problems, and practical resonance in slightly damped problems; mass-spring applications; finding natural frequencies in more general (undamped) conservative systems via conservation of energy equations.*
3.6: 3, 5, 7, 8, 11, 13, 17, 18, **20**, 21, **22** .

w7.1) Consider the 3^{nd} order differential operator for $y(x)$:

$$L(y) := y''' - 3y' + 2y .$$

a) Find the solution space to the homogeneous differential equation $L(y) = 0$. Hint: first find an integer root of the characteristic polynomial, then do long division.
b) Use the method of undetermined coefficients to find a particular solution to $L(y) = x$.
c) Use the method of undetermined coefficients to find a particular solution to $L(y) = e^{3x}$.
d) Use the method of undetermined coefficients to find a particular solution to $L(y) = e^{-2x}$. Hint: this will involve a lot of product rule differentiation if you just plug in the correct undetermined coefficients trial solution. An alternate shortcut you might consider would be to factor L so that $(D + 2I)$ is one of the factors and is the part of L you apply first. We do an example like this on Monday February 23.
e) Use your work in a,b,c,d and linearity (superposition) to write down the general solution to

$$L(y) = 8x + 20e^{3x} - 8e^{-2x}.$$

f) Hand in a technology check (e.g. Wolfram alpha, Maple or Python) for your answer to **e**.

w7.2) Consider problem 3.5.52 from the point of view of undetermined coefficients, but let's take advantage of your knowledge of operator factorization and Euler's formula. We wish to solve

$$L(y) := y'' + 9y = \sin(3x) .$$

Since the roots of the characteristic polynomial $p(r) = r^2 + 9$ are $r = \pm 3i$, the term on the right, $\sin(3x)$ solves the homogeneous DE, and Case II of undetermined coefficients prescribes an undetermined coefficients test solution of the form

$$y_P(x) = x(A\cos(3x) + B\sin(3x)).$$

Carrying this procedure out works to find a particular solution, but is somewhat messy. Rather, factor L

as

$$L = \mathrm{D}^2 + 9\,I = [\mathrm{D} + 3\,i\,I] \circ [\mathrm{D} - 3\,i\,I].$$

Recall from previous homework that $[\mathrm{D} - r\,I] f(x) \mathrm{e}^{r\,x} = f'(x) \mathrm{e}^{r\,x}$.

a) Use the factorization above, and the fact above, to compute that

$$L\left(x\,\mathrm{e}^{3\,i\,x}\right) = 6\,i\,\mathrm{e}^{3\,i\,x}.$$

b) Use Euler's formula and linearity to deduce from the real and imaginary parts of the formula in **a** that

$$L(x\cos(3\,x)) = -6\sin(3\,x)$$
$$L(x\sin(3\,x)) = 6\cos(3\,x).$$

c) Use linearity and the result of **b** to deduce that a particular solution to the original differential equation is $y_P(x) = -\frac{1}{6}x\cos(3\,x)$.

w7.3 Energy in a mass-spring-damper system: Let $x(t)$ be the position of a mass m attached to a spring with Hooke's constant k and damping piston with constant c, yielding the differential equation

$$m\,x'' + c\,x' + k\,x = f(t),$$

where $f(t)$ is an external forcing on the mass. We define the total energy to be the sum of kinetic and potential energy. We wish to study how the total energy decays because of heat loss when there is damping, and how the forcing function $f(t)$ can add or subtract energy to the system. "Recall" from physics that potential energy $PE(t)$ is stored by the compressed or stretched spring, and is the work done to stretch/compress it as the mass moves from the equilibrium $x = 0$, to position x:

$$PE(t) = \int_0^x k u \, \mathrm{d}u = \frac{k}{2}x^2 .$$

As usual, the kinetic energy of the mass is

$$KE(t) = \frac{m}{2}(x')^2.$$

The sum is the total energy

$$E(t) = PE(t) + KE(t) = \frac{1}{2}\left(k\,x(t)^2 + m\,(x'(t))^2\right).$$

a) Take the derivative of $E(t)$ with respect to time. Use the chain rule on the right side of the equation. Use the original differential equation to subsitute for $x''(t)$, and simplify your work so that you get a formula for $E'(t)$ that only depends on the forcing $f(t)$, the velocity $x'(t)$, and the damping coefficient c.

b) Assume there is no external forcing, i.e. $f = 0$. In this case, show that the rate of change in total energy is negative if c is positive and velocity is non-zero. This represents the conversion of kinetic energy into (lost) heat energy. On the other hand, show that if $c = 0$ then total energy stays constant.

c) Set $f = 0$, $m = 1$, $c = 2$, $k = 5$. Set initial conditions $x(0) = 2$ and $x'(0) = 0$. How long will it take for the system to loose 80 % of its initial energy? To solve, first find the solution $x(t)$ to the IVP and use this solution to compute the energy function E explicitly.

d) Plot the energy curve (with technology) and describe its behavior. Explain in words the mechanism that drives the energy picture that you observe, and why the energy is not decreasing at the initial instant $t = 0$, but then decreases more rapidly later on.

The last two extended problems are an excursion through most of the physical phenomena described in section 3.6:

Consider a mass-spring-dashpot system with additional external force $F(t)$ being applied to the mass. In particular, we consider a periodic external force $F(t) = F_0 \cos(\omega t)$. As we know, this system is governed by the following differential equation for the displacement $x(t)$ from equilibrium:

$$m x'' + c x' + k x = F_0 \cos(\omega t) .$$

We will take $m = 1\ kg$, $k = .64 \frac{N}{m}$, $F_0 = 2\ N$ in the following problems. Also, the system will always start at rest, i.e. $x(0) = 0$, $x'(0) = 0$. The damping coefficient c will be modified in different problems, as will the angular frequency ω of the driving force.

w7.4) Consider the configuration above, in the undamped case $c = 0$. In particular consider the initial value problem

$$x'' + .64\, x = 2 \cos(\omega t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

a) What is the "natural" angular frequence ω_0 (for the unforced problem) in this differential equation? Hint, the natural frequency is defined to be the angular frequency for the solutions to the unforced and undamped differential equation, which in this case is the DE

$$x'' + .64\, x = 0.$$

b) Assume $\omega \neq \omega_0$: Use the method of undetermined coefficients to solve for the particular solution $x_P(t)$ for the forced differential equation. Then use $x(t) = x_P(t) + x_H(t)$ to solve the IVP. Check your answer with technology.

c) Write down the special case of the solution in **b** when $\omega = .7$. Compute the period of this solution, which is a superposition of two cosine functions. Use technology to graph one period of the solution. What phenomenon is exhibited by this solution?

d) Solve the IVP when $\omega = \omega_0$. Use the method of undetermined coefficients or operator factorization, i.e. $L = D^2 + .64\, I = [D + .8\, i\, I] \circ [D - .8\, i\, I]$ as in w7.2, to find a particular solution, and then use $x = x_P + x_H$ to solve the IVP. Check your answer with technology. Graph the solution on the interval $0 \leq t \leq 60$ seconds. What phenomenon is exhibited by this solution?

w7.5) Consider the same mass-spring dashpot system as above, except with $c = 2 \frac{kg}{s}$, and with $\omega = 0.8$. This gives us the differential equation

$$x'' + 2\, x' + 0.64\, x = 2 \cos(.8 \cdot t)$$

a) Use the method of undetermined coefficients to find a particular solution $x_P(t)$ to this differential equation.

b) Use your work from **a** and the solution to the corresponding homogeneous equation to write down the

general solution to this differential equation. Identify the "steady periodic" and "transient" parts of this general solution.
c) Use technology to solve the initial value problem for this differential equation above, with $x(0) = 0, x'(0) = 0$. (This will also let you check your work in parts a,b.)
d) Graph, on a single plot, the steady periodic solution from **b** and the solution to the initial value problem in **c**. Choose a time interval so that you can clearly see the convergence of the IVP solution to the steady periodic solution.