Math 2280-002

Week 4-5 concepts and homework, due February 6.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Wednesday quiz will be drawn from all of these concepts and from these or related problems.

2.6: **4**: (postponed from last week)
3.1: 1, **6**, (in 6 use initial values $y(0) = 10, y'(0) = -5$ rather than the ones in the text), **10**, 11, **12**, **14** (In 14 use the initial values $y(1) = 1, y'(1) = 7$ rather than the ones in the text.), 17, **18**, **27**, 33, 39.
3.2: 1, 2, 5, **8**, 11,13, **16**, 21, **25**, 26

Here are two problems that explicitly connect ideas from sections 3.1-3.2 with linear algebra concepts from Math 2270

w4.1) Consider the 3^{rd} order homogeneous linear differential equation for $y(x)$

$$y'''(x) = 0$$

and let W be the solution space.

a) Use successive antidifferentiation to solve this differential equation. Interpret your results using vector space concepts to show that the functions $y_0(x) = 1, y_1(x) = x, y_2(x) = x^2$ are a basis for W. Thus the dimension of W is 3.

b) Show that the functions $z_0(x) = 1, z_1(x) = x - 2, z_2(x) = \frac{1}{2}(x-2)^2$ are also a basis for W. Hint: If you verify that they solve the differential equation and that they're linearly independent, they will automatically span the 3-dimensional solution space and therefore be a basis.

c) Use a linear combination of the solution basis from part b, in order to solve the initial value problem below. Notice how this basis is adapted to initial value problems at $x_0 = 2$, whereas for an IVP at $x_0 = 0$ the basis in a would have been easier to use.

$$y'''(x) = 0$$
$$y(2) = 7$$
$$y'(2) = -13$$
$$y''(2) = 5.$$

w4.2) Consider the three functions

$$y_1(x) = \cos(2x), \quad y_2(x) = \sin(2x), y_3(x) = \sin\left(2x - \frac{\pi}{6}\right).$$

a) Show that all three functions solve the differential equation

$$y'' + 4y = 0.$$

b) The differential equation above is a second order linear homogeneous DE, so the solution space is 2-dimensional. Thus the three functions y_1, y_2, y_3 above must be linearly dependent. Find a linear dependency. (Hint: use a trigonometry addition angle formula.)

c) Explicitly verify that every initial value problem

$$y'' + 4y = 0$$
$$y(0) = b_1$$
$$y'(0) = b_2$$

has a solution of the form $y(x) = c_1\cos(2x) + c_2\sin(2x)$, and that c_1, c_2 are uniquely determined by

b_1, b_2 . (Thus $\cos(2x)$, $\sin(2x)$ are a basis for the solution space of $y'' + 4y = 0$: every solution $y(x)$ has initial values that can be matched with a linear combination of y_1, y_2 , but once the initial values match the solutions must agree by the uniqueness theorem, so y_1, y_2 span the solution space; y_1, y_2 are linearly independent because if $c_1 \cos(2x) + c_2 \sin(2x) = y(x) \equiv 0$ then $y(0) = y'(0) = 0$ so also $c_1 = c_2 = 0$.)

d) Find by inspection, particular solutions $y(x)$ to the two non-homogeneous differential equations

$$y'' + 4y = 28, \qquad y'' + 4y = -16x$$

Hint: one of them could be a constant, the other could be a multiple of x .

e) Use superposition (linearity) and your work from **c,d** to find the general solution to the non-homogeneous differential equation

$$y'' + 4y = 28 - 16x .$$

f) Solve the initial value problem, using your work above:

$$\begin{aligned} y'' + 4y &= 28 - 16x \\ y(0) &= 0 \\ y'(0) &= 0 . \end{aligned}$$

w4.3) Runge-Kutta is based on Simpson's rule for numerical integration. Simpson's rule is based on the fact that for a subinterval $[d, d+h]$ of length h , the parabola $y = p(x)$ which passes through the points (d, y_0) , $\left(d + \frac{h}{2}, y_1\right)$, $(d+h, y_2)$ has integral

$$\int_d^{d+h} p(x)\, dx = \frac{h}{6} \cdot (y_0 + 4y_1 + y_2) .$$

a) The integral approximation above follows from one on the interval $[-1, 1]$ by an affine change of variables. So first consider the interval $[-1, 1]$. We wish to find the parabolic function

$$q(x) = a x^2 + b x + c$$

with unknown parameters *a*, *b*, *c*. We want $q(-1) = y_0$, $q(0) = y_1$, $q(1) = y_2$. This gives 3 equations in 3 unknowns, to find *a*, *b*, *c* in terms of y_0, y_1, y_2, namely

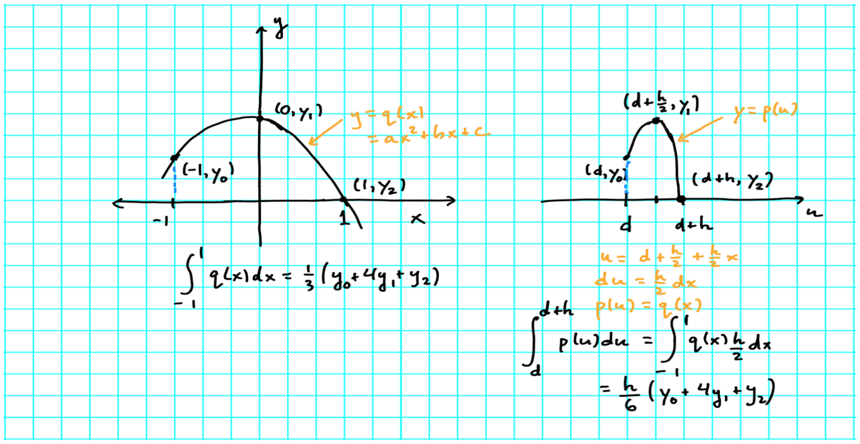
$$\begin{aligned} q(0) &= y_1 = c \\ q(1) &= y_2 = a + b + c \\ q(-1) &= y_0 = a - b + c. \end{aligned}$$

Find *a*, *b*, *c*.

b) Compute $\int_{-1}^{1} q(x)\, dx$ for thes values of *a*, *b*, *c* you find in part a, and verify the equality

$$\int_{-1}^{1} q(x)\, dx = \frac{1}{3}(y_0 + 4y_1 + y_2)$$

Note, the formula for general interval follows from a change of variables, as indicated below:



Remark: If you've forgotten, or if you never talked about Simpson's rule in your Calculus class, here's how it goes: In order to approximate the definite integral of $f(x)$ on the interval $[a, b]$, you subdivide $[a, b]$ into n subintervals of width $\Delta x = \dfrac{b-a}{n} = h$. Then add the midpoints of each subinterval. Label these $x-$ values (incluidng midpoints) as

$$x_0 = a, x_1 = a + \frac{h}{2}, x_2 = a + h, x_3 = x_0 + \frac{3h}{2}, x_4 = x_0 + 2h, \ldots x_{2n} = x_0 + 2nh = b,$$

with corresponding $y-$values $y_i = f(x_i)$, $i = 0,\ldots 2n$. On each successive pair of intervals $[x_{2k}, x_{2k+1}]$ use the parabolic estimate

$$\int_{x_{2k}}^{x_{2k}+h} f(u)\,\mathrm{d}u \approx \frac{h}{6}\cdot\left(f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})\right) = \frac{h}{6}\cdot\left(y_{2k} + 4y_{2k+1} + y_{2k+2}\right)$$

above, estimating the integral of f by the integral of the interpolating parabola on the subinterval. This yields the very accurate (for large enough n) Simpson's rule formula

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{6}\left(\left(y_0 + 4y_1 + y_2\right) + \left(y_2 + 4y_3 + y_4\right) + \ldots + \left(y_{2n-2} + 4y_{2n-1} + y_{2n}\right)\right),$$

i.e.

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6n}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}\right).$$

http://en.wikipedia.org/wiki/Simpson's_rule

w4.4) (Famous numbers revisited, section 2.6, page 135, of text). The mathy numbers e, $\ln(2)$, π can be well-approximated using approximate solutions to differential equations. We illustrate this on Wednesday Jan 30 for e, which is $y(1)$ for the solution to the IVP

$$y'(x) = y$$
$$y(0) = 1.$$

Apply Runge-Kutta with $n = 10, 20, 40 \ldots$ subintervals, successively doubling the number of subintervals until you obtain the target number below - rounded to 8 decimal digits - twice in succession. We will do this in class for e, and you can modify that code if you wish, or use Matlab code that I will post to CANVAS.

a) $\ln(2)$ is $y(2)$, where $y(x)$ solves the IVP

$$y'(x) = \frac{1}{x}$$
$$y(1) = 0$$

(since $y(x) = \ln(x)$).

b) π is $y(1)$, where $y(x)$ solves the IVP

$$y'(x) = \frac{4}{x^2 + 1}$$
$$y(0) = 0$$

(since $y(x) = 4\arctan(x)$ and $\arctan(1) = \frac{\pi}{4}$).

Note that in **a,b** you are actually "just" using Simpson's rule from Calculus, since the right sides of these DE's only depend on the variable x and not on the value of the function $y(x)$. For reference:

```
> Digits := 16 : #In Maple it's easy to specify the number of working digits for calculations.
  evalf(e);  #evaluate the floating point of e
  evalf(π);
  evalf(ln(2));
                              2.718281828459045
                              3.141592653589793
                              0.6931471805599453                    (1)
>
```

w4.5 Consider the following differential equation, which could be modeling the velocity in a linear drag problem for a falling object, if we choose "down" to be the positive direction:

$$v'(t) = 9.8 - .2\,v.$$

a) Find the terminal velocity

b) Solve the IVP for this differential equation with initital condition $v(0) = 0$. Verify that your solution limits to the terminal velocity.

c) Modify the Matlab code posted on CANVAS to do the following:

(i) Find the Euler, Improved Euler, and Runge Kutta numerical approximations to the solution of the IVP with $n = 10$ time steps, on the time interval $0 \le t \le 5$.

(ii) Create a single display which contains the slope field for this differential equation in the box $0 \le t \le 5$, $0 \le v \le 50$; scatter plots of the Euler, Improved Euler and Runge Kutta approximations; and a plot of the actual solution.