Math 2280-002

Week 3-4 concepts and homework, due January 30.

Recall that problems which are not underlined are good for seeing if you can work with the underlying concepts; that the underlined problems are to be handed in; and that the Friday quiz will be drawn from all of these concepts and from these or related problems.

1.5: input-output modeling: **38**

2.2: phase diagram analysis: 5, 7, 9, 11

2.3: improved velocity-acceleration models:
constant, or constant plus linear drag forcing: **2, 3**, **9**, **10** ,12,
quadratic drag: **13, 14,** 17

2.4-2.6: numerical methods for approximating solutions to first order initial value problems.
2.4: **4**: Euler's method
2.5: **4**: improved Euler
2.6: **4**: Runge-Kutta *NOTE: this problem is postponed until next week.*

w3.1) (from 2.2 - this is the postponed problem w2.4 from last week). Consider the autonomous differential equation

$$\frac{dx}{dt} = x^4 - 9x^2 .$$

a) Find the equilibria; draw the phase portrait;
b) classify the equlibria as stable, asymptotically stable, or unstable (possibly one-sided stable);
c) use dfield to sketch the slope field and representative solution graphs, including the graphs of the equilibrium solutions, to verify your phase portrait analysis. Include this plot in your homework.

w3.2 (2.2 applications of population models) Consider a bioreactor used by a yogurt factory to grow the bacteria needed to make yogurt. As long as no bacteria are being harvested, the growth of the bacteria is governed by the logistic equation

$$\frac{dP}{dt} = k \cdot P(M - P)$$

where P is the population in millions and t is the time in days. Recall that M is the carrying capacity of the reactor, and k is a positive constant.
a) Through observation it is found that after a long time the population in the reactor stabilizes at 50 million bacteria, and that when the population of the reactor is 20 million bacteria the population increases at a rate of 12 million per day. From this, find k and M in the governing equation.
b) If the colony starts with a population of 10 million bacteria, how long will it take for the population to reach 80 % of carrying capacity?
c) Assume the reactor allows for continuous harvesting of the bacteria, without shutting down the reactor. Let h be the rate at which the bacteria are harvested, in millions per day. Write down the new differential equation governing the bacteria population. What is the maximum rate of harvesting h that will not cause the population of bacteria to go extinct? (Harvesting at less than this rate will ensure that there is always a stable equilibrium point where P is positive.) Hint: This question is all about phase diagram analysis.

w3.3 (2.3 improved velocity-acceleration models) Suppose you wish to dispose of nuclear waste by placing it in sealed drums and dropping the drums into the ocean. Each filled drum weighs 1280 lb (so has a mass of $\frac{1280}{32}$ = 40 slugs), and has a volume of 18 cubic feet. If we choose "up" to be the positive position direction, then the velocity differential equation for a drum falling though water is:

$$m \frac{dv}{dt} = -W + B + F_R$$

where W is the weight of the drum, B is the buoyant force equal to the weight of the water displaced by the drum (the weight density of water is 62.5 pounds per cubic foot), and F_R is the force of water resistance, known to be 1 lb for every $\frac{ft}{s}$ that the drum is moving. The drums may burst if they hit the ocean floor at a speed higher than 120 $\frac{ft}{s}$.

Note: A technology check is expected for computations below. Any claimed numeric answer, symbolic answer or equation should be verified. If a technology answer check is impossible, then provide on paper whatever details are possible.

a) Assuming the drums are dropped into the ocean with $v_0 = 0$, find an expression for the velocity $v(t)$ ($g = 32 \frac{ft}{s^2}$).

b) Find the time t at which the drum velocity has magnitude is 120 ft/s.

c) What is the deepest water into which the drums can be dropped without violating the bursting speed of 120 ft/s?