Math 2280-002
Week 12 concepts and homework
5.6, 9.1
Due Wednesday April 10

5.6 *Fundamental matrices and matrix exponentials.*

1, **3**, 11, **23**, **25**. In 1, 3 you are using equation (8), below. For those two problems also exhibit e^{tA}, which can be computed by $\Phi(t)\Phi(0)^{-1}$.

THEOREM 1 Fundamental Matrix Solutions

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0. \tag{8}$$

THEOREM 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0, \tag{27}$$

and this solution is unique.

Thus the solution of homogeneous linear systems reduces to the task of computing exponential matrices. Conversely, if we already know a fundamental matrix $\mathbf{\Phi}(t)$ for the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then the facts that $e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{C}$ (by Eq. (4′)) and $e^{\mathbf{A}\cdot 0} = e^{\mathbf{0}} = \mathbf{I}$ (the identity matrix) yield

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}. \tag{28}$$

5.7 *Nonhomogeneous linear systems and the integrating factor method of solutions.*
1, 15, 17 (**w12.2** below).

w12.1 Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

a Compute e^{tA} directly from the power series definition of matrix exponentials. Hint: To make your final answer look compact use the hyperbolic trig functions

$$\cosh(t) := \frac{1}{2}\left(e^t + e^{-t}\right) = \sum_{n\ even;\ n=0}^{\infty} \frac{t^n}{n!}$$

$$\sinh(t) := \frac{1}{2}\left(e^t - e^{-t}\right) = \sum_{n\ odd;\ n=1}^{\infty} \frac{t^n}{n!}.$$

b Recompute e^{tA} using the eigendata of A, and the identity which follows from diagonalization, namely that for P the invertible matrix whose columns are a basis for $\mathbb{R}^n$, and D the diagonal matrix of eigenvalues,

$$e^{tA} = P\, e^{tD} P^{-1} = \Phi(t)\Phi(0)^{-1}.$$

c In this example the first order system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

corresponds to the second order differential equation (with negative Hooke's constant - watch your solutions grow!)

$$x''(t) - x(t) = 0$$

with the usual correspondence. See e.g. question 2 on our last exam. So the columns of e^{tA} should be $[x(t), x'(t)]^T$ for the solutions to this second order DE, with initial conditions $[x(0), x'(0)] = [1, 0]^T$, $[0, 1]^T$, respectively. Verify that this is the case. So for this example that would have been a third way to find the matrix exponential - solve the second order IVP's and make those the columns of e^{tA}.

<u>w12.2</u> In the previous homework problem **<u>w8.4</u>** you solved the following system of differential equations, which arose from a two-tank input-output model for which solute was flowing into and out of the system:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 60 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In order to practice with the integrating factor method of solving nonhomogeneous sytems of linear differential equations with constant matrix coefficients, re-find the solution using the formula we derive in class on Friday: For the system IVP

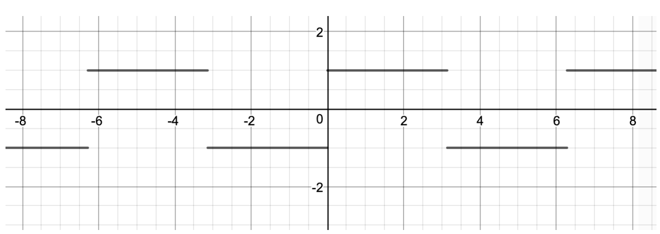
$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}} + \underline{\mathbf{f}}(t)$$
$$\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0$$

The solution is

$$\underline{\mathbf{x}}(t) = e^{tA}\underline{\mathbf{x}}_0 + \int_0^t e^{(t-s)A}\underline{\mathbf{f}}(s)\, ds\ .$$

w12.3) Define the "square wave" function $sq(t)$ to be the 2π-periodic extension of

$$f(t) = \begin{cases} 1 & 0 \le t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$



a) Compute the Fourier series for $sq(t)$. Hint: your answer should be a sine series, since $sq(t)$ is an odd function. Also, it should look dubious that the series converges, since if you tested for absolute convergence you'd get something that looks close to the divergent harmonic series. But theorems actually guarantee that the Fourier series for this function converges at each t-value. (At the jump points it converges to zero, which is the average of the left and right-hand limits that the theorem says. In this case it's because each sine term evaluates to zero at integer multiples of π.)

b) Create a plot of the truncated Fourier series - including just the first five nonzero terms in the Fourier series, for the interval $-8 \le t \le 8$. Hints: Your graph should approximate the exact graph, above; and Desmos lets you enter summation formulas if you want to save time typing. Hand in a printout of your graph.