Math 2280-002
Week 11-12 concepts and homework
section 5.4
Due Wednesday April 3

5.4) Second order systems of differential equations arising from conservative systems. Identifying fundamental modes and natural angular frequencies; forced oscillation problems and the potential for practical resonance when the forcing frequency is close to a natural frequency.
5.4: **2** ,3, **8**, **12**, 13, 14, 16, 18 .

w11.1) This is a continuation of **2, 8**. Now let's force the spring system in problem 2, with a sinusoidal force on the first mass at (variable) angular frequency ω, as in the slightly different text example on pages 330-331. Thus we consider the system

$$\begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + F_0 \cos(\omega t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

a) Find a particular solution of the form

$$\underline{\boldsymbol{x}}_P(t) = \cos(\omega t)\underline{\boldsymbol{c}}\ .$$

Hint: Plug this guess into the differential equation. You will notice that each term simplifies to some vector times the function $\cos(\omega t)$. Thus, after you factor out the $\cos(\omega t)$ term you are left with a matrix equation to solve for $\underline{c} = \underline{c}(\omega)$. You will get formulas analogous to equations (34, 35) in section 5.4, except your c_1, c_2 will blow up at $\omega = 1, 3$, the natural frequencies for this problem.

b) The general solution to this forced oscillation problem is the particular solution from part (a), plus the general solution to the homogeneous problem, which you found in **5.4.2**. Continue assuming that $\omega \neq 1, 3$, and write down this general solution in this case.

c) In a physical problem with a slight amount of damping but the same masses and spring constants, the particular solution would be close to the one you found in part (a), and the homogeneous solutions would be close to the ones you found in **5.4.2**, except that they would be (slowly) exponentially decaying because of the damping. Thus the particular solution would be the steady periodic solution, and the homogeneous solution would be transient. By plotting the magnitude $\|\underline{c}(\omega)\| = \sqrt{c_1(\omega)^2 + c_2(\omega)^2}$ for the undamped problem as a function of ω, you create a "practical resonance" chart analogous to those we created in Chapter 5 without having to actually solve the case of exact resonance. Create such a plot, for the angular frequency range $0 \le \omega \le 5$. Use $F_0 = 4$. Your plot should look like Figure 5.4.10, except your magnitude function will peak at $\omega = 1, 3$.

d) An alternative to finding homogeneous and general solutions for second order systems of the form

$$\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}} + \underline{\mathbf{f}}(t)$$

in the case where A is diagonalizable is to make the change of vector functions to $\underline{\mathbf{u}}(t)$, where P is a matrix whose columns are an eigenbasis for $\mathbb{R}^2$ and

$$\underline{\mathbf{x}}(t) = P\,\underline{\mathbf{u}}(t).$$

Then, in analogy to the discussion for the same substitution techniques in first order systems, and in particular your previous homework problem w8.4 in homework 8, we get a system that decouples, for $u_1(t), u_2(t)$:

$$P\,\underline{\mathbf{u}}''(t) = A\,P\,\underline{\mathbf{u}} + \underline{\mathbf{f}}(t)$$
$$\underline{\mathbf{u}}''(t) = P^{-1}AP\,\underline{\mathbf{u}} + P^{-1}\underline{\mathbf{f}} = D\,\underline{\mathbf{u}} + P^{-1}\underline{\mathbf{f}}$$

where D is the diagonal matrix of eigenvalues.

Re-find the general solution in **b**, using this method. You are still assuming $\omega \neq 1, 3$, but notice that if you used the Chapter 3 formulas for how to solve forced resonance problems you would even be able to treat those cases with this method. What you will be using in our case is that the general solution to the scalar differential equation

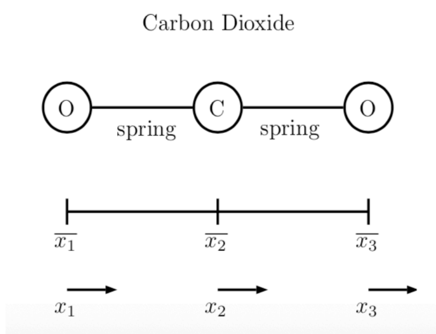
$$u''(t) + \omega_0^2\,u(t) = F_0\cos(\omega\,t)$$

is

$$u(t) = \frac{F_0}{\omega_0^2 - \omega^2}\cos(\omega\,t) + c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

when $\omega \neq \omega_0$. (I reproduce that formula here so you don't have to re-find it, although you certainly could.)

w11.2) In this problem we want to investigate the vibrational properties of carbon dioxide, CO_2, where we use linearization of the real forces to model the interactions between atoms as a linear spring model. We'll focus on longitudinal as opposed to transverse vibrations, in which case the model looks as follows:



We will choose units of mass and time so that the masses of the carbon and oxygen atoms are $m_C = 6$, $m_O = 8$, which are each half of their atomic weights; and so that the spring constant $k = 48$ (so that the eigenvalues of the associated acceleration matrix will be integers).

a) Use Newton's second law to derive the system of three second order differential equations for the displacements $x_1(t), x_2(t), x_3(t)$ of the masses from their equilibrium locations.

b) Rewrite the system in part **a** in the form

$$\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}$$

where $\underline{\mathbf{x}}(t)$ is the vector of displacements and A is the "acceleration" matrix. (Hint: Mathematically this is analogous to the three-car train in Example 2 of section 5.4).

c) Find the eigenvalues and eigenspaces of the acceleration matrix in **b**. Do this work by hand, but feel free to check your answer with technology.

d) Write down the general solution to the system in part **b**, and use it to describe the two fundamental modes of longitudinal vibration and the one fundamental mode of translation for the longitudinal motions of CO_2. (We'll discuss this sort of behavior for a 2-car train in class. See also Theorem 1 on page 325 of the text.) You can see the two longitudinal vibration fundamental modes illustrated, along with two transverse oscillation modes, at the website

http://www.chemtube3d.com/vibrationsCO2.htm