Math 2280-002
Week 10-11 concepts and homework Due Wednesday March 27
sections 6.3-6.4

Interacting population models
6.3: **11**, **12**, **13**, 18, **19**, 20, **22**, 23, 24, **25**, **32**. (In 32, use pplane)

nonlinear mechanical models
6.4: **12**, 13, **14**, 15, **16**.

w10.1 By using separation of variables as we do in class on Friday and as the text does in section 6.3, show that the first quadrant equilbrium in **6.3.19** really is a stable center for the nonlinear problem.

w10.2 Consider the undamped pendulum system of section 6.4 and class notes, for $[\theta(t), \theta'(t)]^T$:

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x)$$

Show that the separation of variables method

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = -\frac{g}{L}\frac{\sin(x)}{y}$$

reproduces the fact that we begin with in class, namely that solution trajectories stay on level curves for the total energy function. This is the reason that the configurations with the mass at the bottom of the pendulum really are stable centers for the undamped rigid rod pendulum, as we discuss in class and as the text also discusses.

w10.3 (extra credit) Recall the competition model for two species $x(t), y(t)$:

$$x'(t) = a_1 x - b_1 x^2 - c_1 x y$$
$$y'(t) = a_1 y - b_2 y^2 - c_2 x y$$

where the constants $a_1, a_2, b_1, b_2, c_1, c_2$ are all positive. We mentioned the theorem that if the logistic inhibition, as measured by the product $b_1 b_2$ is greater than the competitive pressure as measured by $c_1 c_2$, i.e. $b_1 b_2 > c_1 c_2$, and if there is a first quadrant equilibrium solution (x_E, y_E), the all solutions with $x_0, y_0 > 0$ converge to (x_E, y_E) as $t \to \infty$. And conversely, if $b_1 b_2 < c_1 c_2$, and if there is a first quadrant equilibrium solution then it is a (unstable) saddle, and if $x_0, y_0 > 0$ then one of the populations will die out, depending on which side of the saddle's separatrices the initial point lies. A key step in this theorem is to understand the linearization of the system at (x_E, y_E).

a) Show that any equibrium solutions which are not on either the x or y axes (or the origin) satisfy the linear system of equations

$$\begin{bmatrix} b_1 & c_1 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

b) Find the Jacobian matrix at such a point (x_E, y_E), and then simplify it using the equation system in part **a**. The answer you're seeking to verify is:

$$J_{@(x_E, y_E)} = \begin{bmatrix} -b_1 x_E & -c_1 x_E \\ -c_2 y_E & -b_2 y_E \end{bmatrix}$$

c) Compute the characteristic polynomial for the Jacobian in **b**. Your answer should be

$$p(\lambda) = \lambda^2 + \lambda(b_1 x_E + b_2 y_E) + (b_1 b_2 - c_1 c_2) x_E y_E$$

d) Notice that $p(\lambda)$ is a concave up parabola; that $p(0)$ changes sign depending on whether $b_1 b_2 < c_1 c_2$ or $b_1 b_2 > c_1 c_2$; that $p'(0) > 0$ because $x_E, y_E > 0$. Use this information and the quadratic formula to deduce that

$b_1 b_2 < c_1 c_2$ implies (x_E, y_E) is a saddle point.

$b_1 b_2 > c_1 c_2$ implies (x_E, y_E) is either a nodal sink or a spiral sink.