Wed Feb 6:
3.3 Solving constant coefficient homogeneous linear differential equations

Friday !!

Announcements:

Warm-up Exercise:

For the next two sections we focus homogneous linear differential equations with constant coefficients. Section 3.3 contains the algorithms we'll need and in section 3.4 we'll apply the general theory to the unforced mass-spring differential equation.

3.3: Algorithms for the basis and general (homogeneous) solution to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are all constant.

step 1) Try to find a basis for the solution space made of exponential functions....try $y(x) = e^{rx}$. In this case

$$L(y) = e^{rx}\left(r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0\right) = e^{rx} p(r) .$$

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$ if we want. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, \ldots, r_n$, then

$$e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$$

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x} .$$

Example: The differential equation

on Tuesday.

$$y''' + 3y'' - y' - 3y = 0$$

has characteristic polynomial

$$p(r) = r^3 + 3r^2 - r - 3 = (r+3)\cdot\left(r^2 - 1\right) = (r+3)(r+1)\cdot(r-1)$$

so the general solution to

$$y''' + 3y'' - y' - 3y = 0$$

is

$$y_H(x) = \underset{y_1}{c_1 e^{x}} + \underset{y_2}{c_2 e^{-x}} + \underset{y_3}{c_3 e^{-3x}} .$$

to check linear ind (& span) for soln space.

Check $W(y_1, y_2, y_3) = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} e^x & e^{-x} & e^{-3x} \\ e^x & -e^{-x} & -3e^{-3x} \\ e^x & e^{-x} & 9e^{-3x} \end{bmatrix}$

has inverse at some x_0 $(x_0 = 0)$

@ $x=0$, compute det...

Exercise 1) By construction, $e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n-dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < \dots < r_n$ and to use a limiting argument.

skip...

Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, \ldots r_m$ $(m < n)$ with the r_j all different, but some of the factors $(r - r_j)$ in $p(r)$ appear with powers bigger than 1 . In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \ldots (r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \ldots + k_m = n$.

Start with a small example: The case of a second order DE for which the characteristic polynomial has a double root.

Exercise 2) Let r_1 be any real number. Consider the homogeneous DE

$$L(y) := y'' - 2 r_1 y' + r_1^2 y = 0 .$$

with $p(r) = r^2 - 2 r_1 r + r_1^2 = (r - r_1)^2$, i.e. r_1 is a double root for $p(r)$. Show that $e^{r_1 x}, x e^{r_1 x}$ are a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is $y_H(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$. Start by checking that $x e^{r_1 x}$ actually (magically?) solves the DE. (We may wish to study a special case $y'' + 6 y' + 9 y = 0$.)

HW 3.1.10

$L(y) = y'' - 10y' + 25y = 0$

try $y = e^{rx}$

$L(y) = e^{rx}\left[\underbrace{r^2 - 10r + 25}\right]$

$p(r) = (r-5)^2$.

text said $y_1 = e^{5x}$

$y_2 = xe^{5x}$

and you checked it.

Seems liked magic but it's not.

See HW.

Here's the general algorithm: If

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2} \ldots (r - r_m)^{k_m}$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}.$$

This yields k_j solutions for each root r_j, so since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation. There's a good explanation in the text as to why these additional functions actually do solve the differential equation, see pages 316-318 and the discussion of "polynomial differential operators". I've also made a homework problem in which you can explore these ideas. Using the limiting method we discussed earlier, it's not too hard to show that all n of these solutions are indeed linearly independent, so they are in fact a basis for the solution space to $L(y) = 0$.

illustrates general case.

Exercise 3) Explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$L(y) = y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in Case 2 above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H. When you compare to the repeated roots algorithm, note that it includes the possibility $r = 0$ and that $e^{0x} = 1$.

$$y^{(4)} - y^{(3)} = 0$$
$$e^{-x}\left(y^{(4)} - y^{(3)}\right) = 0$$
$$\frac{d}{dx}\left(e^{-x} y'''\right) = 0$$
$$e^{-x} y''' = C$$
$$y''' = Ce^x$$
$$y'' = Ce^x + D$$
$$y' = Ce^x + Dx + E$$
$$y = Ce^x + \frac{D}{2}x^2 + Ex + F$$

$y = c_1 e^x + c_2 x^2 + c_3 x + c_4$

basis is $\{e^x, 1, x, x^2\}$

comparison

$p(r) = r^4 - r^3$
$= r^3(r-1)$
$= (r-0)^3(r-1)$

recipe: $\{e^{0x}, xe^{0x}, x^2e^{0x}, e^x\}$
$\{1, x, x^2, e^x\}$.

Case 3) Complex number roots - this will be our surprising and fun topic on Friday. Our analysis will explain exactly how and why trig functions and mixed exponential-trig-polynomial functions show up as solutions for some of the homogeneous DE's you worked with in your homework and lab for this past week. This analysis depends on Euler's formula, one of the most beautiful and useful formulas in mathematics:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
for $i^2 = -1$.

$p(r) = (r+2)^3(r-1)^2$ for order 5 DE
roots $r = -2, 1$ recipe for basis $\{e^{-2x}, xe^{-2x}, x^2e^{-2x}, e^x, xe^x\}$

Fri Feb 8

3.3 Solving constant coefficient homogeneous linear differential equations: complex roots in the characteristic polynomial

Announcements: • we're a day behind.

Warm-up Exercise: In w4.2 HW you studied the DE for $y(x)$

$$L(y) := y'' + 4y = 0$$

and showed that $\{\cos 2x, \sin 2x\}$ was a basis for the solution space, i.e.

$$y_H(x) = c_1 \cos 2x + c_2 \sin 2x$$

On the other hand, what happens when you look for exponential functions $y(x) = e^{rx}$ that satisfy this DE, as you did for other HW problems ?? i.e. what is $L(e^{rx})$?

$$L(e^{rx}) = e^{rx}\underbrace{(r^2+4)}_{(r-2i)(r+2i)} = 0$$

roots of $p(r)$ are

$$r = \pm 2i$$

turns out e^{2ix}, e^{-2ix} are solns

Euler's formula:
$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{i(2x)} = \cos 2x + i\sin 2x$$
$$e^{i(-2x)} = \cos 2x - i\sin 2x$$

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$
for homog sol fns

3.3 continued. How to find the solution space for n^{th} order linear homogeneous DE's with constant coefficients, and why the algorithms work.

Strategy: In all cases we first try to find a basis for the n-dimensional solution space made of or related to exponential functions....trying $y(x) = e^{r\,x}$ yields

$$L(y) = e^{r\,x}\left(r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0\right) = e^{r\,x} p(r)\,.$$

The characteristic polynomial $p(r)$ and how it factors are the keys to finding the solution space to $L(y) = 0$. There are three cases, of which the first two (distinct and repeated real roots) are in ~~yesterday~~'s Wednesday's notes.

Case 3) $p(r)$ has complex number roots. This is the hardest, but also most interesting case.The punch line is that exponential functions $e^{r\,x}$ still work, except that $r = a \pm b\,i$; but, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions.

To understand how this all comes about, we need to learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions actually are important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

- $$f(x) \sim f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + ... + \frac{1}{n!}f^{(n)}(0)x^n + ...$$

(Recall that the partial sum polynomial through order n matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also needed error estimates to figure out on which intervals the Taylor polynomials actually coverged back to f.)

Exercise 1) Use the formula above to recall the three very important Taylor series for

$f(x) = e^x$, $f'(x) = e^x$, $f^{(n)}(x) = e^x$ @ $x = 0$, $f^{(n)}(0) = e^0 = 1$

1a) $e^x =$ $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

1b) $\cos(x) =$

$f(x) = \cos x = f^{(4)}$ | @ $x=0$: 1

$f'(x) = -\sin x = f^{(5)}$ | 0

$\vdots$

$f''(x) = -\cos x$ | -1

$f'''(x) = \sin x$ | 0

$f^{(4)}(x) = \cos x$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

1c) $\sin(x) =$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

In Calculus you checked that these series actually converge and equal the given functions, for all real numbers x.

Exercise 2) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta)\ .$$

$i^2 = -1$

$i^3 = i^2 i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!}$$

$\cos\theta$

$+\ i\sin\theta$