3b) Show that every IVP

$$y'' - 2y' - 3y = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

✓ can be solved with a unique linear combination $y(x) = c_1 \underset{\substack{\| \\ e^{3x}}}{y_1(x)} + c_2 \underset{\substack{\| \\ e^{-x}}}{y_2(x)}$, (where c_1, c_2 depend on b_0, b_1).

Then use the uniqueness theorem to deduce that y_1, y_2 span the solution space to this homogeneous differential equation. Since these two functions are not constant multiples of each other, they are linearly independent and a basis for the *2-dimensional* solution space!

$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{-x}$$
$$y'(x) = c_1 y_1' + c_2 y_2' = 3c_1 e^{3x} + -c_2 e^{-x}$$

$$\begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} = \underset{\substack{\uparrow \\ \text{Wronskian} \\ \text{matrix of } y_1, y_2 \\ W(y_1,y_2)}}{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{3x} & e^{-x} \\ 3e^{3x} & -e^{-x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

its determinant is called "the Wronskian"

want to find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

@ $x = 0$:

$$\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = W(y_1,y_2)(0) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underset{\substack{\uparrow \\ \det = -4}}{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underset{\substack{\uparrow \\ \frac{1}{-4}\begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix}}}{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}^{-1}} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

Showed for each $\begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$ there's a unique $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ to solve IVP.

This makes $\{y_1(x), y_2(x)\}$ a basis for the solution space.

- $y_1(x) = e^{3x}$, $y_2(x) = e^{-x}$ span the sol. space.

Because: Let $y(x)$ solve the DE.

Let $\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$. Find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ so that $c_1 e^{3x} + c_2 e^{-x}$ has the same initial value vector.

So by the uniqueness thm (each IVP only has a single sol'n),
It must be that $c_1 e^{3x} + c_2 e^{-x} \equiv y(x)$!!

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is always 2-dimensional on any interval I for which the hypotheses of the existence-uniqueness theorem hold.

We'll see why this is always true, tomorrow.

- y_1, y_2 linearly ind.

Let: $c_1 y_1 + c_2 y_2 \equiv 0$

$\Rightarrow c_1 y_1' + c_2 y_2' \equiv 0$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

@ $x=0$: $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ shows linear ind.

Note: By showing the Wronskian matrix of e^{3x}, e^{-x}, at $x=0$, is invertible (i.e. its det $\neq 0$)
we conclude both that $\{e^{3x}, e^{-x}\}$ span sol'n space. And that they're linearly independent.
i.e. a basis

Tues Feb 5:

3.1-3.2 Second order and n^{th} order linear differential equations, and vector space theory connections.

Announcements:
- I'll be in the lab 2:00-3:00 after class
- continue on theory/methods for 2^{nd} order linear DE's then breeze through n^{th} order $(n \geq 2)$, "the same"

12:56

Warm-up Exercise: factor $p(r) = 1 \cdot r^3 + 3r^2 - r - 3$?! hint: all roots are integers

$= (r+1)(r+3)(r-1)$ ✓.

roots must divide -3

$\pm 1, \pm 3$

$r = -1, -3, 1.$

e.g. $p(1) = 1 + 3 - 1 - 3 = \underline{\underline{0}}$

so $(r-1)$ factor.

$$\begin{array}{r|l} & r^2 + 4r + 3 \\ \hline r-1 & r^3 + 3r^2 - r - 3 \\ & -(r^3 - r^2) \\ \hline & 4r^2 - r - 3 \\ & -(4r^2 - 4r) \\ \hline & 3r - 3 \\ & (3r - 3) \\ \hline & 0 \end{array}$$

✓.

$r^3 + 3r^2 - r - 3$

$= (r-1)(r^2 + 4r + 3)$

$= (r-1)(r+3)(r+1)$

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional on any interval I for which the hypotheses of the existence-uniqueness theorem hold.

proof:

Pick any $x_0 \in I$. Find solutions $y_1(x), y_2(x)$ to initial value problems at x_0 so that the so-called Wronskian matrix for y_1, y_2 at x_0

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible (i.e. $\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}$ are a basis for $\mathbb{R}^2$, or equivalently so that the determinant of the Wronskian matrix (called just the Wronskian) is non-zero at x_0).

- You may be able to find suitable y_1, y_2 by a method like we used in the last example on Monday, but the existence-uniqueness theorem guarantees they exist even if you don't know how to find formulas for them.

Under these conditions, the solutions y_1, y_2 are actually a basis for the solution space! Here's why:

• span: the condition that the Wronskian matrix is invertible at x_0 means we can solve each IVP there with a linear combination $y = c_1 y_1 + c_2 y_2$: In that case, to solve the IVP

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y(x_0) &= b_0 \\ y'(x_0) &= b_1 \end{aligned}$$

we set

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

At x_0 we wish to find c_1, c_2 so that

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= b_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= b_1. \end{aligned}$$

This system is equivalent to the the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

When the Wronskian matrix at x_0 has an inverse, the unique solution $[c_1, c_2]^T$ is given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Since the uniqueness theorem says each IVP has a unique solution, we've found it!

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

- Span: Since each solution $y(x)$ to the differential equation solves *some* initial value problem at x_0, this gives all solutions, as we let $[b_0, b_1]^T$ vary freely in $\mathbb{R}^2$. So each solution $y(x)$ *is* a linear combination of y_1, y_2. Thus $\{y_1, y_2\}$ spans the solution space.

- Linear independence: If we have the identity

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

then by differentiating each side with respect to x we also have

$$c_1 y_1{}'(x) + c_2 y_2{}'(x) = 0.$$

Evaluating at $x = x_0$ this is the system

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= 0 \\ c_1 y_1{}'(x_0) + c_2 y_2{}'(x_0) &= 0 \end{aligned}$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Theorem 4: All solutions to the <u>nonhomogeneous</u> second order linear DE

$$L(y) = y'' + p(x)y' + q(x)y = f(x)$$

are of the form $y = y_P + y_H$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE.

proof: Make use of the fact that

$$L(y) := y'' + p(x)y' + q(x)y$$

is a linear operator. In other words, use the *linearity properties*

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

① Let $L(y_p) = f$ (just saying that y_p is a single particular soln)
Let $L(y_H) = 0$ (" " " y_H " " homog. soln)

then $y_p + y_H$ also solves $L(y) = f$
because $L(y_p + y_H) = L(y_p) + L(y_H)$ because L is linear
$= f + 0 = f$

② Let y_Q be another particular soln
$L(y_Q) = f$ also

write $y_Q = y_p + \underbrace{(y_Q - y_p)}$

$L(y_Q + (-y_p)) = L(y_Q) + L(-y_p)$
$= L(y_Q) - L(y_p)$
$= f - f = 0$

showed $y_Q = y_p + y_H$
where $L(y_H) = 0$!

In Monday's notes we found that the general solution to the homogeneous differential equation

$$y'' - 2y' - 3y = 0$$

is

$$y_H = c_1 e^{-x} + c_2 e^{3x}.$$

Now consider the non-homogeneous differential equation

$$y'' - 2y' - 3y = 6.$$

Notice that

$$\boxed{y_P = -2}$$

is one particular solution to the differential equation. (If we'd guessed that there might be a constant solution, we could've substituted $y(x) \equiv d$ into the differential equation and deduced that $d = 2$.)

Exercise 1a) Solve the initial value problem

$$y'' - 2y' - 3y = 6.$$
$$y(0) = -1$$
$$y'(0) = -5$$

with a solution to the differential equation of the form

$$y = y_P + y_H = -2 + c_1 e^{-x} + c_2 e^{3x}.$$

$y'' - 2y' - 3y = 0.$

$y_H(x) = c_1 e^{-x} + c_2 e^{3x}$

($y_c(x)$ in text)

so $y'(x) = y_P' + y_H' = 0 - c_1 e^{-x} + 3c_2 e^{3x}$

want

$$@\ x=0:\quad \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \end{bmatrix}}_{\begin{bmatrix} y_P(0) \\ y_P'(0) \end{bmatrix}} + \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}}_{W(e^{-x}, e^{3x})(0)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix} \quad \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

1b) Notice that the same algebra shows you could solve every initial value problem

solvable solvable

$$y'' - 2y' - 3y = 6.$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

with a solution of the form

$$y = y_P + y_H = -2 + c_1 e^{-x} + c_2 e^{3x}$$

so by the uniqueness theorem for initial value problems, these ALL the solutions to the differential equation even though we did not get them a direct method like we used for first order linear differential equations.

The theory for n^{th} order linear differential equations is conceptually the same as for second order...

Definition: An n^{th} order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots + A_1(x)y' + A_0(x)y = F(x)\ .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A_n(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$

($a_{n-1}, \ldots a_1, a_0, f$ are all functions of x, and the DE above means that equality holds for all value of x in the interval I.)

Theorem 1 (Existence-Uniqueness Theorem): Let $a_{n-1}(x), a_{n-2}(x), \ldots a_1(x), a_0(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$\begin{gathered} y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ y''(x_0) = b_2 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{gathered}$$

and $y(x)$ exists and is n times continuously differentiable on the entire interval I.

The differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = f$$

is called <u>linear</u> because the operator L defined by

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y$$

satisfies the so-called <u>linearity properties</u>

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

• *The proof that L satisfies the linearity proporties is just the same as it was for the case when n* = 2, which we checked.

The following two theorems only use the linearity properties of the operator L. I've kept the same numbering we used for the case $n = 2$.

<u>Theorem 2:</u> The solution space to the homogeneous linear DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

is a subspace.

<u>Theorem 4:</u> The general solution to the <u>nonhomogeneous</u> n^{th} order linear DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = f$$

is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

Theorem 3: The solution space to the n^{th} order homogeneous linear differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y \equiv 0$$

is n-dimensional. Thus, any n independent solutions $y_1, y_2, \ldots y_n$ will be a basis, and all homogeneous solutions will be uniquely expressible as linear combinations

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

proof: By the existence half of Theorem 1, we know that there are solutions for each possible initial value problem for this (homogenenous case) of the IVP for n^{th} order linear DEs. So, pick solutions $y_1(x), y_2(x), \ldots\, y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$\begin{bmatrix} y_1(x_0) \\ y_1{}'(x_0) \\ y_1{}''(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2{}'(x_0) \\ y_2{}''(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix}, \ldots, \begin{bmatrix} y_n(x_0) \\ y_n{}'(x_0) \\ y_n{}''(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}$$

are a basis for $\mathbb{R}^n$ (i.e. these n vectors are linearly independent and span $\mathbb{R}^n$. (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

Claim: In this case, the solutions $y_1, y_2, \ldots y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these n functions and the dimension of the solution space is n .... discussion on next page.

- Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the Wronskian matrix at x_0 times the vector of linear combination coefficients:

$$\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

has a unique solution $\underline{\mathbf{c}}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$y(x) = z(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Thus $y_1, y_2, \ldots y_n$ <u>span</u> the solution space.

- **<u>linear independence</u>**: If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then differentiate this identity $n - 1$ times, and then substitute $x = x_0$ into the resulting n equations. This yields the Wronskian matrix equation above, with $\left[b_0, b_1, \ldots b_{n-1}\right]^T = [0, 0, \ldots, 0]^T$. So the matrix equation above implies that $\left[c_1, c_2, \ldots c_n\right]^T = \underline{\mathbf{0}}$. So $y_1, y_2, \ldots y_n$ are also <u>linearly independent</u>.

- Thus $y_1, y_2, \ldots y_n$ are a <u>basis</u> for the solution space and the general solution to the homogeneous DE can be written as

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Let's do some new exercises that tie these ideas together. (We may do these exercises while or before we wade through the general discussions on the previous pages!)

Exercise 2) Consider the 3^{rd} order linear homogeneous DE for $y(x)$:

$$L(y) = y''' + 3y'' - y' - 3y = 0.$$

Find a basis for the 3-dimensional solution space, and the general solution. Use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

try $y = e^{rx}$

$$L(y) = e^{rx}\left[r^3 + 3r^2 - r - 3\right]$$

$$= e^{rx}\left[(r+3)(r-1)(r+1)\right] \equiv 0 \qquad r = -3, 1, -1$$

$$y_1(x) = e^{-3x}, \quad y_2(x) = e^{x}, \quad y_3(x) = e^{-x}$$