Math 2280-002
Week 5: February 4-8
3.1-3.3 linear differential equations of arbitrary order.

Mon Feb 4:
3.1-3.2 Second order linear differential equations and vector space theory connections, continued

In Chapter 3 we'll be using vector space theory to understand solutions to differential equations!

Announcements: • I'll be in tutoring ctr lab for at least half an hour after class

12:57

Warm-up Exercise: Find constants c_1, c_2 to solve the IVP for $y(x)$

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = 3 \\ y'(0) = 1 \end{cases}$$

$$y(x) = c_1 e^{3x} + c_2 e^{-x}$$

We verified that these functions all solve the DE, on Friday.

answer $y(x) = e^{3x} + 2e^{-x}$

(check: $y(0) = 1 + 2 = 3$ ✓
$y'(0) = 3 - 2 = 1$ ✓)

how

$$y(x) = c_1 e^{3x} + c_2 e^{-x}$$
$$y'(x) = 3c_1 e^{3x} - c_2 e^{-x}$$

@ $x = 0$

$$y(0) = c_1 + c_2 = 3$$
$$y'(0) = 3c_1 - c_2 = 1$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{-4}\begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -4 \\ -8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} ✓$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

when A^{-1} exists

$$|A| = ad - bc$$

Directions for Matlab homework (or the open source and free GNU Octave)

1) If you're working on Math Department system computers and unless you've changed your login information, and if your name is

<u>J</u>ane <u>**Q**</u>. <u>**P**</u>ubli<u>**C**</u>

your login name is made from the location of the letters that are capitalized above, and is of the form

c-pcjq (or c-pcjq1, c-pcjq2, etc. if there are multiple students with your initials).
If you don't have a middle initial the last letter is omitted.

If the last four digits of your UID are 4397 then unless you've changed your password, it is

pcjq4397

2) Open a Browser (e.g. Firefox in the Math lab) and download the files in the "numerics" directory on our homework page (or on CANVAS) and save them to a directory on your computer. The URL is

http://www.math.utah.edu/~korevaar/2280spring19/numerics/

3) Find Matlab on your computer and open it.

4) From Matlab find the directory you created with our class matlab files.
Modify the "class-example.m" file (or copy/paste/modify pieces of it into a new ".m" file in order to complete the homework problem w4.5.
Modify the "famous_numbers.m" file in order to complete the homework problem w4.4
Change comments to make them appropriate to your work.

5) Please hand in hard copies of the output that is asked for in w4.4 and w4.5, along with printouts of the two scripts you wrote that generate the output.

6) If you or a friend can't figure out how to do something in Matlab, use google to ask about what it is you're trying to do. For example, if you're curious about plotting you could query "how to make plots in matlab", "how to create a matlab display with several plots", etc.
Google will lead you to Matlab help directories, or to forums where other people have asked similar questions and received answers.

There are a number of video introductions to the Matlab environment on youtube.
Just search for something like "youtube introduction to matlab".

The two main goals in Chapter 3 are

(1) to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the corresponding initial value problems with n initial values:

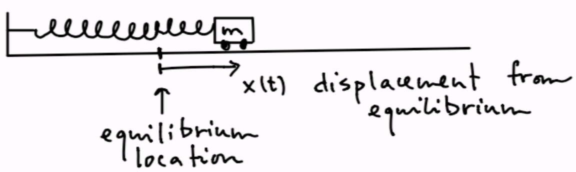
$$
\begin{gathered}
y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \\
y(x_0) = b_0 \\
y'(x_0) = b_1 \\
y''(x_0) = b_2 \\
\vdots \\
y^{(n-1)}(x_0) = b_{n-1}
\end{gathered}
$$

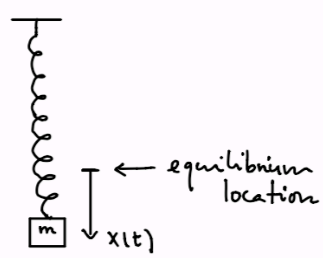
<u>and</u>

(2) to learn important physics/engineering applications of these general techniques. The applications we learn about in this course will be for second order linear differential equations ($n = 2$), as below.

<u>Applications Example</u> (sections 3.4 and 3.6): The forced-damped-oscillator differential equation. Here $x(t)$ is the function, instead of $y(x)$, as often happens in our textbook when we move between theory and applications. This particular differential equation arises in a multitude of contexts besides just the mass-spring model, as we shall see.

$$
\begin{gathered}
m\,x'' + c\,x' + k\,x = F(t) \,. \\
x(0) = x_0 \\
x'(0) = v_0
\end{gathered}
$$





Newton's 2^{nd} law:

$$
\begin{aligned}
m\,x''(t) &= \text{Net forces} \\
&= 0 \quad - c\,x' \quad - kx \quad + F(t)
\end{aligned}
$$

0: net forces at equilibrium, i.e. when $x, x' \equiv 0$

$-cx'$: linear drag

$-kx$: linear "Hookes' law" force from stretching or compressing spring

$F(t)$: possible "external" forcing function

On Friday we started discussing second order linear differential equations - we'll see that differential equations with (higher) order n follow the same conceptual outline that we began on Friday for $n = 2$.

Definition: A general *second order linear differential equation* for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y'' + p(x)y' + q(x)y = f(x) .$$

analogous to first order linear differential equations in Chapters 1-2:

$$y' + p(x)y = q(x).$$

Exercise 1) Find all solutions to to second order differential equation for $y(x)$

$$y'' + 2y' = 0$$

on the x-interval $-\infty < x < \infty$.

1^{st} order for fun $y'(x)$, $P(x) = 2$, I.F. e^{2x}

$$e^{2x}\left[y'' + 2y'\right] = 0e^{2x} = 0$$

$$\frac{d}{dx}\left[e^{2x}y'\right] = 0$$

$$e^{2x}y' = C$$

$$y'(x) = Ce^{-2x}$$

$$y(x) = -\frac{C}{2}e^{-2x} + D$$

$$y(x) = c_1 e^{-2x} + c_2$$

two free parameters, just like in warmup.

Theorem 1 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$\text{IVP} \quad \begin{cases} y'' + p(x)y' + q(x)y = f(x) \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \end{cases}$$

we'll explain why in Chapter 4.

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Example For the forced mass-spring model this would be saying that once you specify the initial displacement and velocity of the mass, and given the values of m, c, k and the known forcing function, the future motion of the mass is uniquely determined .... i.e. the experiment is repeatable with the same result. This make intuitive sense.

Exercise 2) Verify Theorem 1 for the interval $I = (-\infty, \infty)$ and the IVP

$$\begin{cases} y'' + 2y' = 0 \\ y(0) = b_0 \\ y'(0) = b_1 \end{cases}$$

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$$y(x) = c_1 e^{-2x} + c_2$$
$$y'(x) = -2c_1 e^{-2x}$$

(i.e. $y \in \text{span}\{y_1, y_2\}$, $y_1(x) = e^{-2x}$, $y_2(x) = 1$)

to solve IVP

$$y(0) = b_0 = c_1 + c_2$$
$$y'(0) = b_1 = -2c_1$$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

i.e. $$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}^{-1}}_{\frac{1}{2}\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

The real reason that differential equations of the form

$$y'' + p(x)y' + q(x)y = f(x)$$

are called *linear* is that the "linear operator" L that operates on functions by the rule

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called *linear transformation properties*

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

(Recall that the matrix multiplication function $L(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ satisfies the analogous properties) Any time we have have a transformation L satisfying (1),(2), we say it is a *linear transformation.* We touched on general linear transformations in Math 2270, linear algebra.

Example: If

$$L(y) := y'' + 2\,y',$$

and

$$y_1(x) = x^2,\ y_2(x) = e^{3x},\ y_3(x) = e^{-2x}$$

we can compute at x,

$$(x^2)'' + 2(x^2)'$$

$$L(y_1)(x) = 2 + 2\cdot 2\,x = 2 + 4\,x$$

$$L(y_2)(x) = 9\,e^{3x} + 2\cdot 3\,e^{3x} = 15\,e^{3x}$$

$$L(y_3)(x) = 4\,e^{-2x} + 2\cdot \cancel{2}^{(-2)}\,e^{-2x} = 0. \qquad \text{(We knew that!)}$$

$$x^2 + e^{3x}$$

$$L(y_1 + y_2)(x) = (2 + 9\,e^{3x}) + 2(2\,x + 3\,e^{3x})$$
$$= L(y_1)(x) + L(y_2)(x) \ !$$

$$L(5\,y_1)(x) = (10 + 2\cdot 10\,x) = 5\,L(y_1)(x).$$

On Friday we checked the linearity properties (1),(2) for the general second order differential operator L and general functions $y_1(x), y_2(x)$. In other words we showed that the operator L defined by

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called *linearity properties*

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

Here's how the checking went:

$$L(y_1 + y_2) = (y_1 + y_2)'' + p(x)(y_1 + y_2)' + q(x)(y_1 + y_2)$$

$$= y_1'' + y_2'' + p(x)(y_1' + y_2') + q(x)\cdot(y_1 + y_2)$$

$$L(y_1 + y_2) = (y_1'' + p\,y_1' + q\,y_1) + (y_2'' + p\,y_2' + q\,y_2).$$

$$L(c\,y) = (c\,y)'' + p\,(c\,y)' + q\,(c\,y)$$

$$= c\,y'' + c\,p\,y' + c\,q\,y$$

$$= c\,(y'' + p\,y' + q\,y)$$

$$L(c\,y) = c\,L(y).$$

Note that by applying (1), (2) repeatedly or by following the same procedure,

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2).$$

$$L(c_1 y_1 + c_2 y_2 + c_3 y_3) = c_1 L(y_1) + c_2 L(y_2) + c_3 L(y_3) \text{ etc.}$$

In other words, our linear differential operator (and any *linear transformation* from Math 2270*)* transforms linear combinations of inputs into linear combinations of the outputs, with the same *weights,* c_1, c_2, c_3 ...

Sub vector space ...

In Math 2270 we saw that for any linear transformation $T : V \to W$,

$$ker\ T = \{v \in V \mid T(v) = 0\}$$

is a subspace. A special case of this was *Nul A* for matrix transformations $T(x) = A\,x$. For function vector spaces your class may or may not have discussed the derivative operator, D, as an example of a linear transformation, i.e. $\mathrm{D}(y) := y'$.

Theorem 2: the solution space to the *homogeneous* second order linear DE

$$L(y) := y'' + p(x)y' + q(x)y = 0$$

is a subspace. Notice that this is analogous to the proof we used in Math 2270 to show *Nul A* is a subspace, and is a special case of the general fact about *kerT* for linear transformations *T*.

a subspace is closed under addition and scalar multiplican

proof α) closed under +.

If $L(y_1) = 0$ and $L(y_2) = 0$

then $L(y_1 + y_2) = L(y_1) + L(y_2) = 0 + 0 = 0.$

so sums of homog. sols are homog sols.

β) closed under scalar mult:

If $L(y_1) = 0$, $c \in \mathbb{R}$

then $L(cy_1) = cL(y_1) = c \cdot 0 = 0.$

(so also, linear combos of soltns are soltns)

Unlike what happened in Exercises 1, 2, and unlike what is true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing, as in the following example:

Exercise 3) Consider the homogeneous linear DE for $y(x)$

$$L(y) = y'' - 2y' - 3y = 0$$

knew from Friday & warmup that e^{-x}, e^{3x} solve this DE.

3a) Find two exponential functions $y_1(x) = e^{rx}, y_2(x) = e^{\rho x}$ that solve this DE. (Recall Friday warmup exercise in this context.)

guess e^{rx}, see what r needs to be.

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$$L(y) = r^2 e^{rx} - 2(re^{rx}) - 3e^{rx}$$

$$= e^{rx}\underbrace{\left[r^2 - 2r - 3\right]}_{(r-3)(r+1)}$$

$L(y) = 0$ if $r = 3$ $(e^{3x} = y_1)$

or $r = -1$ $(e^{-x} = y_2)$

because homog linear DE, soltn space is subspace, so

$y(x) = c_1 e^{3x} + c_2 e^{-x}$ also soltns.

3b) Show that every IVP

$$y'' - 2y' - 3y = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$, (where c_1 , c_2 depend on b_0 , b_1).

Then use the uniqueness theorem to deduce that y_1, y_2 span the solution space to this homogeneous differential equation. Since these two functions are not constant multiples of each other, they are linearly independent and a basis for the *2-dimensional* solution space!

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is always 2-dimensional on any interval I for which the hypotheses of the existence-uniqueness theorem hold.

We'll see why this is always true, tomorrow.