Wed Feb 27
• Chapters 4-6 systems of differential eqtns

4.1 Introduction to systems of differential equations

Announcements:

• 1 pointed out on next HW when we'll talk about related examples
• Quiz today on § 3.6 : Forced oscillations
  three important physical phenomena
  • beating
  • resonance (pure)
  • practical resonance

$mx'' + cx' + kx = F_0 \cos \omega t$

Warm-up Exercise: !! Find the eigenvalues and eigenvectors (eigenspace bases) for

$$A = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$$

$\lambda = 0, -6$

$E_{\lambda=0} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

✓ check $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$(A\vec{v} = \lambda\vec{v})$.

$E_{\lambda=-6} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

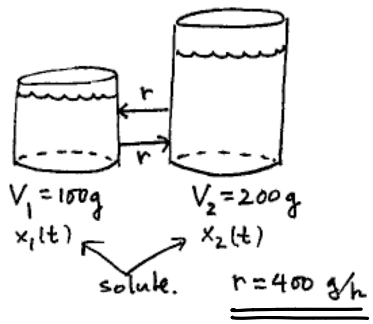
✓ check $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} = -6\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4.1 Systems of differential equations - to model multi-component systems via compartmental analysis

http://en.wikipedia.org/wiki/Multi-compartment_model

and to understand higher order differential equations.

Here's a relatively simple 2-tank problem to illustrate the ideas:



Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$x_1'(t) = r_i c_i - r_o c_o \qquad r_i = r_o = 400$

$x_1'(t) = 400 \frac{x_2}{200} - 400 \frac{x_1}{100} = -4x_1 + 2x_2$

$\frac{gal}{hour} \quad \frac{lb}{gal}$

$x_2'(t) = r_i c_i - r_o c_o$

$= 400 \frac{x_1}{100} - 400 \frac{x_2}{200}$

$x_2'(t) = 4x_1 - 2x_2$

A in warmup.

$$\boxed{\begin{aligned} x_1'(t) &= -4x_1 + 2x_2 \\ x_2'(t) &= 4x_1 - 2x_2 \end{aligned}}$$

$x_1(0) = b_1$
$x_2(0) = b_2$

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The example on page 1 is a special case of the general <u>initial value problem for a first order system of differential equations</u>:

$\vec{x}(t)$: $\vec{x} : [a, b] \to \mathbb{R}^n$

$$\text{IVP} \quad \begin{cases} \underline{\mathbf{x}}'(t) = \underline{\mathbf{F}}(t, \underline{\mathbf{x}}(t)) \\ \underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0 \end{cases}$$

Chapter 1

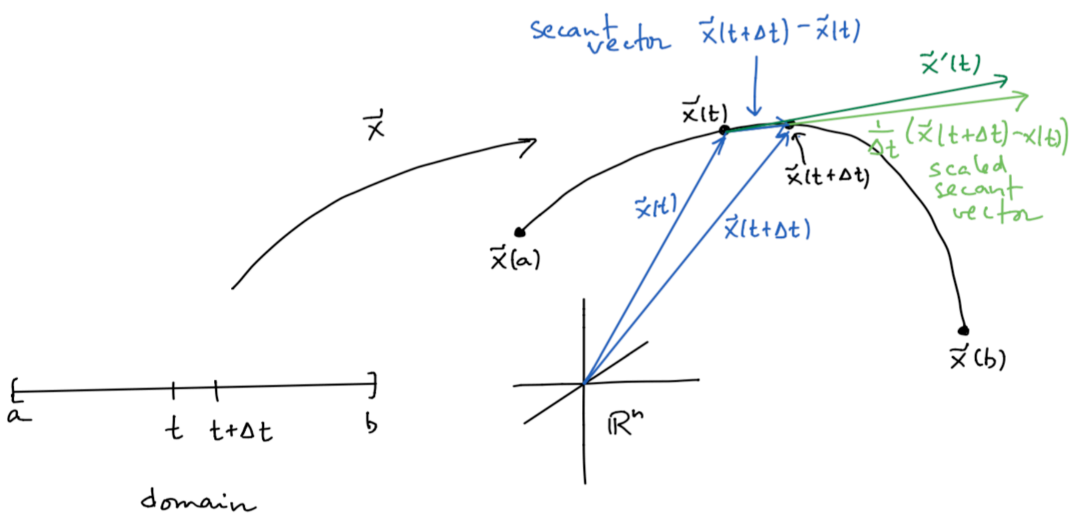
$$\begin{cases} x'(t) = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

<u>Existence and Uniqueness for solutions to IVP's</u>: The IVP above is a vectorized version of the scalar first order DE IVP that we considered in Chapter 1. In Chapter 1 we understood why (with the right conditions on the right hand side), these IVP's have unique solutions. There is an analogous existence-uniqueness theorem for the vectorized version we study in Chapters 4-6, and it's believable for the same reasons the Chapter 1 theorem seemed reasonable. We just have to remember the geometric meaning of the *tangent* vector $\underline{\mathbf{x}}'(t)$ to a parametric curve in $\mathbb{R}^n$ (which is also called the *velocity* vector in physics, when you study particle motion):

Algebra:

$$\underline{\mathbf{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ \vdots \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \right) = \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

Geometric interpretation in terms of displacement vectors along a parametric curve:



So the existence-uniquess theorem for first order systems of DE's is true because if you know where you start at time t_0, namely $\underline{\mathbf{x}}_0$; and if you know your tangent vector $\underline{\mathbf{x}}'(t)$ at every later time -in terms of your location $\underline{\mathbf{x}}(t)$ and what time t it is, as specified by the vector function $\underline{\mathbf{F}}(t, \underline{\mathbf{x}}(t))$; then there should only be one way the parametric curve $\underline{\mathbf{x}}(t)$ can develop. This is analogous to our reasoning in Chapter 1 that there should only be one way to follow a slope field, given the initial point one starts at.

Exercise 2) Return to the page 1 tank example

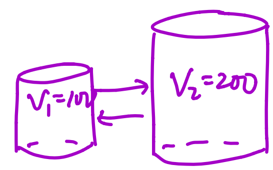
$$x_1'(t) = -4x_1 + 2x_2$$
$$x_2'(t) = 4x_1 - 2x_2$$
$$x_1(0) = 9 \quad \leftarrow \text{all solute starts in tank 1}$$
$$x_2(0) = 0$$

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" (tangent vector) vector field (which is time-independent in this example), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank, at time t. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

2b) What are the apparent limiting solute amounts in each tank?

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?

2b) $\lim_{t\to\infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$



2c) because total salt is 9 lb & eventually concentration should be equal everywhere & 2nd tank is twice as large

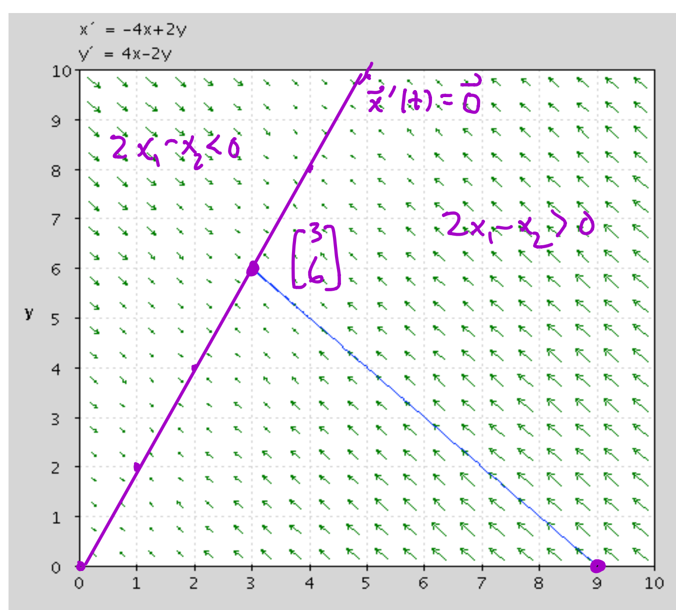
pplane $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4x_1 - 2x_2 \\ 4x_1 - 2x_2 \end{bmatrix} =$

$= (4x_1 - 2x_2)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= 2(2x_1 - x_2)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\vec{x}'(t) = \vec{0}$ when $2x_1 - x_2 = 0$

$x_2 = 2x_1$



$\begin{bmatrix} 9 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

<u>Definition</u>: Any first order system of differential equations which can be written in the form

$$\underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$

is called a *first order linear system of DE's*. (Here $\underline{\boldsymbol{x}}(t)$ and $\underline{\boldsymbol{f}}(t)$ are functions from an interval in $\mathbb{R}$, with range lying in $\mathbb{R}^n$, and $P(t)$ is an $n \times n$ matrix whose entries are functions of t. For us $P(t)$ will almost always be a constant matrix. If the system can be written in the form

$$\underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$$

we say that the linear system of differential equations is *homogeneous*. Otherwise it is *non-homogeneous* or *inhomogeneous*.

Notice that the operator on vector-valued functions $\underline{\boldsymbol{x}}(t)$ defined by

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}}(t)$$

is <u>linear</u>, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{y}}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{\boldsymbol{y}}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t)).$$

SO! The space of solutions to the homogeneous first order system of differential equations

$$\underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$$

is a subspace. AND the general solution to the inhomogeneous system

$$\underline{\boldsymbol{x}}'(t) + P(t)\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$

will be of the form

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$$

where $\underline{\boldsymbol{x}}_P$ is any single particular solution and $\underline{\boldsymbol{x}}_H$ is the general homogeneous solution.

In the case that $P(t) = -A$ is a constant matrix (i.e. entries don't depend on t) , we usually write the homogeneous system as

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}\,.$$

In the case that A is a diagonalizable matrix it turns out we can always find a basis for the homogeneous solution space made of vector-valued functions of the form

$$\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}\,,$$

where $\underline{\boldsymbol{v}}$ an eigenvector of A and λ is its eigenvalue, i.e.

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}.$$

System of DE's:

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

Candidate solution:

$$\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}},$$

where $\underline{\boldsymbol{v}}$ an eigenvector of A and λ is its eigenvalue, i.e.

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}.$$

We can verify that such an $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ solves the homogeneous DE system above by showing we get a true identity when we substitute it in. We compute the left side of the differential equation:

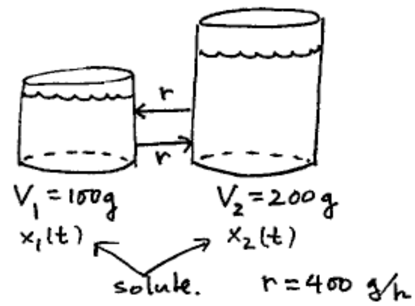
$$\underline{\boldsymbol{x}}(t) = e^{\lambda t} v \quad \Rightarrow \underline{\boldsymbol{x}}'(t) = \lambda\, e^{\lambda t}\underline{\boldsymbol{v}}.$$

And we compute the right side

$$A\,\underline{\boldsymbol{x}}(t) = A\, e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t} A\,\underline{\boldsymbol{v}} = e^{\lambda t}\lambda\,\underline{\boldsymbol{v}}.$$

Same!

Exercise 3) Use the eigendata of the matrix in our running example solve the initial value problem of Exercise 2!! Compare your solution $\underline{\mathbf{x}}(t)$ to the parametric curve drawn by pplane.



$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$