Tues Feb 26

3.6-3.7: Forced oscillations: practical resonance; 3.7 Electrical circuits analog

Announcements: studying

yesterday: beating $\omega \approx \omega_0$, $\omega \neq \omega_0$; resonance $\omega = \omega_0$ $\quad x''(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos \omega t$

today: practical resonance $\quad x''(t) + c x' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$

when c "small", $\omega \approx \omega_0$

Homework questions?

Warm-up Exercise: Find a particular solution x(t) to

$$x'' + 2x' + 26x = 82\cos 4t$$

$$\begin{aligned}
26\,(x_p &= d_1\cos 4t + d_2 \sin 4t\,)\\
+\,2\,(x_p' &= -4d_1 \sin 4t + 4 d_2 \cos 4t\,)\\
+\,1\,(x_p'' &= -16 d_1 \cos 4t - 16 d_2 \sin 4t\,)
\end{aligned}$$

$$\begin{aligned}
L(x_p) &= \cos 4t\,(26d_1 + 8d_2 - 16d_1) \\
&\quad + \sin 4t\,(26 d_2 - 8d_1 - 16 d_2)
\end{aligned}
\quad \underset{\text{want}}{=} \quad \cos 4t\,(82) + \sin 4t\,(0)$$

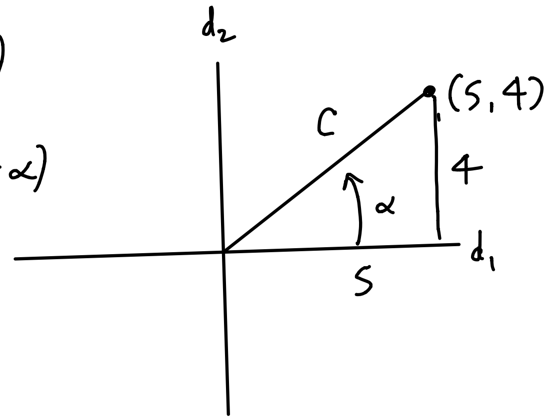
$$\begin{aligned}
10d_1 + 8d_2 &= 82\\
-8d_1 + 10d_2 &= 0
\end{aligned}$$

$\div 2$

$$\begin{bmatrix} 5 & 4 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2\end{bmatrix} = \begin{bmatrix} 41 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} d_1 \\ d_2\end{bmatrix} = \frac{1}{\cancel{41}}\begin{bmatrix} 5 & -4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} \cancel{41} \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{aligned}
x_p(t) &= 5\cos 4t + 4\sin 4t\\
&= C\cos(4t-\alpha)\\
&= \sqrt{41}\cos(4t-\alpha)
\end{aligned}$$

$$\alpha = \arctan\left(\tfrac{4}{5}\right)$$



<u>Damped forced oscillations</u> $(c > 0)$ for $x(t)$:

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_P(t)$:

$$k\left[x_P = A\cos(\omega t) + B\sin(\omega t)\right]$$
$$+\, c\left[x_P{}' = -A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t)\right]$$
$$+\, m\left[x_P{}'' = -A\,\omega^2\cos(\omega t) - B\,\omega^2\sin(\omega t)\right] .$$

$$L(x_P) = \cos(\omega t)\left(k A + c B\omega - m A\omega^2\right) + \sin(\omega t)\left(k B - c A\omega - m B\omega^2\right) .$$

$\underset{=}{\text{want}}$ $\cos\omega t\,(F_0) + \sin\omega t\,(0)$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\omega^2 & c\,\omega \\ -c\,\omega & k - m\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \overbrace{\frac{1}{\left(k - m\omega^2\right)^2 + c^2\omega^2}\begin{bmatrix} k - m\omega^2 & -c\,\omega \\ c\,\omega & k - m\omega^2 \end{bmatrix}}^{\text{inverse}}\begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\omega^2\right)^2 + c^2\omega^2}\begin{bmatrix} k - m\omega^2 \\ c\,\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

with

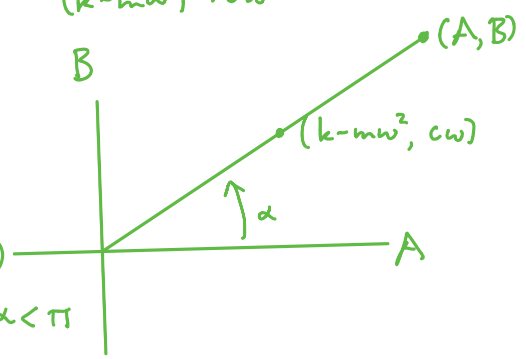
$$C = \left\|\begin{bmatrix} A \\ B \end{bmatrix}\right\| = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\sqrt{(k-m\omega^2)^2 + c^2\omega^2}$$

$$C = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}} \quad . \text{ (Check!)}$$

$$\cos(\alpha) = \frac{k - m\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}} > 0$$

$0 < \alpha < \pi$



And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p t}C_1\cos(\omega_1 t - \alpha_1)$.
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p t}(c_1 t + c_2)$.
- over-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

<u>Important to note:</u>

- The amplitude C in x_{sp} can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is <u>practical resonance</u>.
- The phase angle α is always in the first or second quadrant.

From previous page:

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

Since

$$k - m\omega^2 = m\left(\omega_0^2 - \omega^2\right)$$

we may rewrite the steady periodic trig data as

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} = \frac{F_0}{m}\,\frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{c^2}{m^2}\omega^2}}\,.$$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{c^2}{m^2}\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} = \frac{c\,\omega}{m\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{c^2}{m^2}\omega^2}}\,.$$

$$k - m\,\omega^2 = m\left(\omega_0^2 - \omega^2\right).$$

Exercise 1) (a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient c! Namely, for c small, when $\omega^2 << \omega_0^2$ we have $\cos(\alpha) \approx 1$, i.e. $\alpha \approx 0$ (in phase with the forcing function) for x_{sp}; when $\omega^2 >> \omega_0^2$ we have $\cos(\alpha) \approx -1$, i.e. α near π (out of phase with the forcing function); for $\omega \approx \omega_0$, $\sin(\alpha) \approx 1$, i.e. $\alpha \approx \frac{\pi}{2}$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

Exercise 2) Find $x_{sp}(t)$ for the forced oscillation problem

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0\,.$$

did this!

Practical resonance: The steady periodic amplitude C for damped forced oscillations is

$$C(\omega) = C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} = \frac{F_0}{m}\,\frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{c^2}{m^2}\omega^2}}\,.$$

Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k} = \frac{F_0}{m\,\omega_0^2}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$. (Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in the variable ω^2 it will have at most one minimum in the variable ω^2, so $C(\omega)$ will have at most one maximum for non-negative ω. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 3a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c: $2x'$

$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t).$$

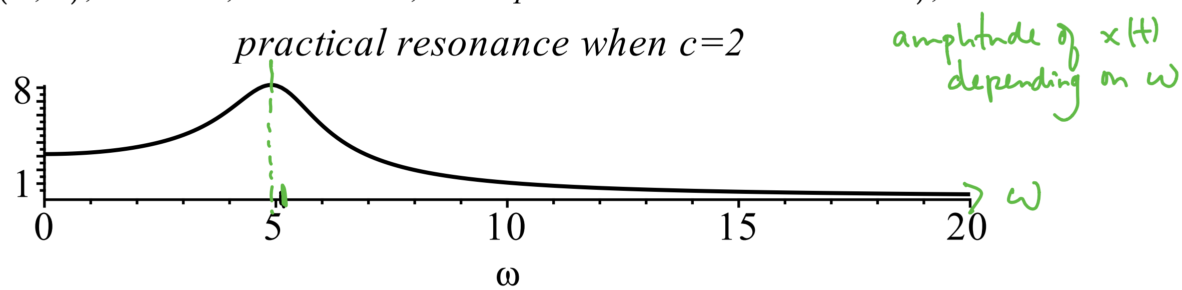
3b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

$$C(\omega) = 82\,\frac{1}{\sqrt{(26-\omega^2)^2 + 4\omega^2}}$$

```
> restart :
> with(plots) :
> C := (ω, c) → 82 / sqrt((26 − ω^2)^2 + c^2·ω^2) :
```

$\omega_0 = \sqrt{26} \approx 5$

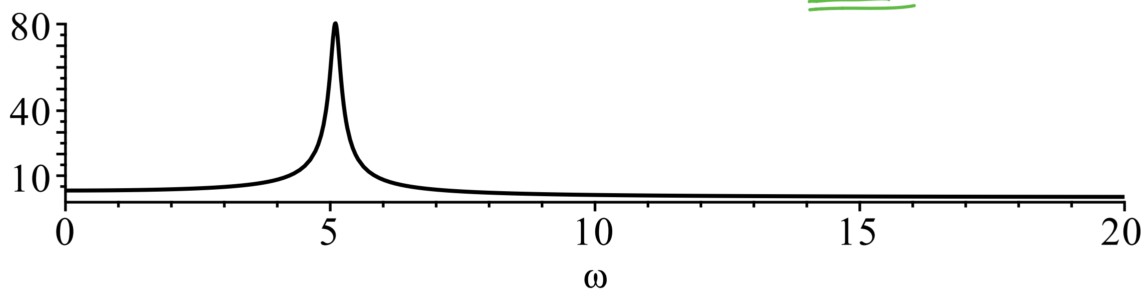
```
> plot(C(ω, 2), ω = 0 ..20, color = black, title = `practical resonance when c=2`);
```




```
>
```

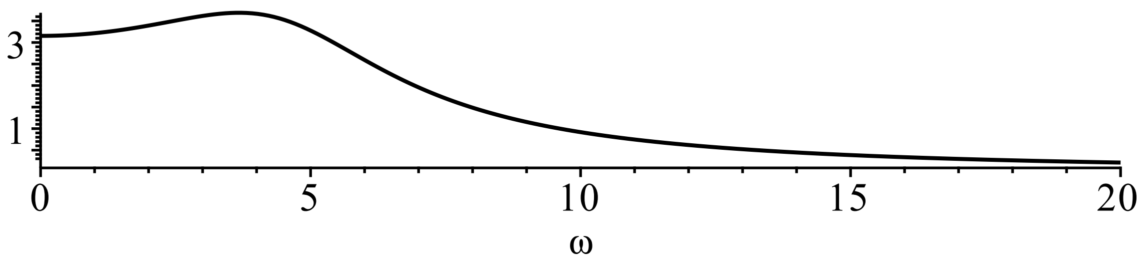
```
> plot(C(ω, .2), ω = 0 ..20, color = black, title = `serious practical resonance when c=0.2`);
```

serious practical resonance when c=0.2



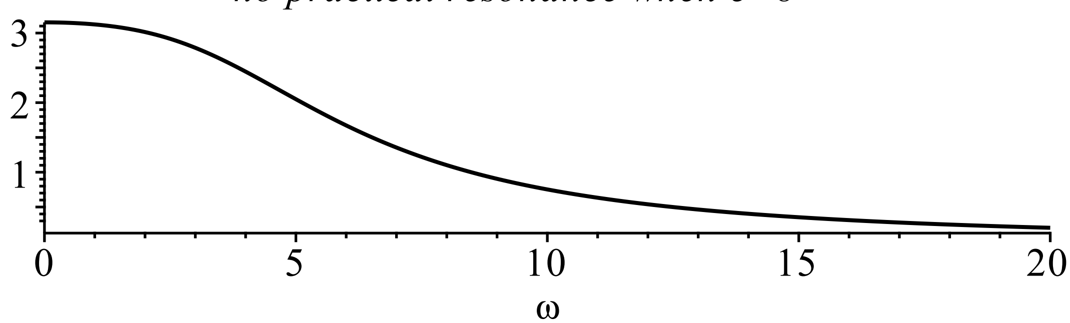

```
> plot(C(ω, 5), ω = 0 ..20, color = black, title = `barely practical resonance when c=5`);
```

barely practical resonance when c=5



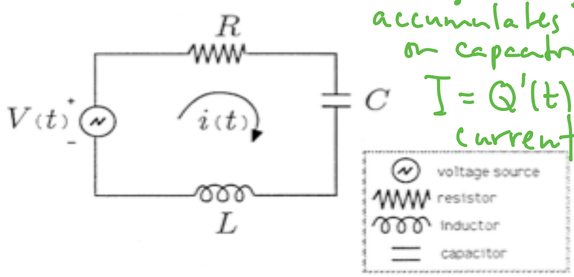

```
> plot(C(ω, 8), ω = 0 ..20, color = black, title = `no practical resonance when c=8`);
```

no practical resonance when c=8



Section 3.7 The mechanical-electrical analogy: Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....the classical RLC circuit with applied voltage is described in this schematic:

charge $Q(t)$ accumulates on capacitor
$I = Q'(t)$ current



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

volts

For $Q(t)$: $$L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t) = E_0 \sin(\omega t) = 120 \sin(60 \cdot 2\pi t)$$

60 Hertz $\left(\frac{\text{cycles}}{\text{sec}}\right)$

For $I(t)$: $$L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t) = E_0\,\omega \cos(\omega t)\ .$$

We can transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + I_{tr}(t)\ .$$

$$I_{sp}(t) = I_0 \cos(\omega t - \alpha) = I_0 \sin(\omega t - \gamma)\ ,\quad \gamma = \alpha - \frac{\pi}{2}\ .$$

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\,\omega^2)^2 + c^2\omega^2}} \quad \Rightarrow I_0(\omega) = \frac{E_0\omega}{\sqrt{\left(\frac{1}{C} - L\,\omega^2\right)^2 + R^2\,\omega^2}}$$

$$\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}}\ .$$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the impedance $Z(\omega)$ of the circuit (because the larger the *impedance*, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\frac{1}{C\,\omega} = L\,\omega$, i.e.

$$C = \frac{1}{L\,\omega^2} \quad \text{if } L \text{ is fixed and } C \text{ is adjustable (old analog radios).}$$

$$L = \frac{1}{C\,\omega^2} \quad \text{if } C \text{ is fixed and } L \text{ is adjustable}$$

Both L and C are adjusted in this M.I.T. lab demonstration:

http://www.youtube.com/watch?v=ZYgFuUl9_Vs.