Math 2280-002
Week 8: 3.6 -3.7, 4.1-4.2
Mon Feb 25
3.6: Forced oscillations

Announcements: today & tomorrow: study important physics phenomena related to forced oscillation problems (§3.6)

Warm-up Exercise: What is your undetermined coefficients guess for $x_p(t)$?

a) $x''(t) + 4x(t) = 5\cos 3t$

$x_p(t) = d_1 \cos 3t$
(don't need $\sin 3t$ terms since only even derivs in L)

b) $x''(t) + 4x(t) = 5\cos 2t$
$x_H(t) = \mathrm{span}\{\cos 2t, \sin 2t\}$

$x_p(t) = d_1 t\cos 2t + d_2 t\sin 2t$

c) $x''(t) + 2x'(t) + 4x(t) = 5\cos 2t$

$x_p(t) = d_1 \cos 2t + d_2 \sin 2t$

Section 3.6: forced oscillations in mechanical systems (and as we shall see in section 3.7, also in electrical circuits) overview:
We study solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\, t)$$

using section 3.5 undetermined coefficients algorithms.

- undamped $(c = 0)$:
  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\, x = 0$$

example on Friday

  which has simple harmonic motion solutions

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C_0 \cos(\omega_0 t - \alpha)\,.$$

So for the non-homongeneous DE the section 5.5 method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}}$ $\Rightarrow x_P = A\cos(\omega\, t)$ because only even derivatives, we don't need $\sin(\omega\, t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega\, t) + C_0 \cos(\omega_0 t - \alpha_0)\,.$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0, A \approx C_0$ Beating! ← interesting phenomenon
- $\omega = \omega_0$ case 2 section 3.5 undetermined coefficients; since

$$p(r) = r^2 + \omega_0^2 = (r + i\omega_0)^1 (r - i\omega_0)^1$$

x

our undetermined coefficients guess is

$$x_P = t^1\left(A\cos(\omega_0\, t) + B\sin(\omega_0\, t)\right)$$
$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega\, t - \alpha) + C_0 \cos(\omega_0 t - \alpha_0)\,.$$

("pure" resonance!)

part (b) on warmup.

linearly growing amplitude
← interesting phenomenon.

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega\, t) + B\sin(\omega\, t) = C\cos(\omega\, t - \alpha)$ (because the roots of the characteristic polynomial are never purely imagninary $\pm\, i\,\omega$ when $c > 0$).
  - underdamped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + e^{-p\,t} C_1 \cos(\omega_1 t - \alpha_1)\,.$
  - critically-damped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + e^{-p\,t}(c_1 t + c_2)\,.$
  - over-damped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\,.$

part (c) warmup

- in all three damped cases on the previous page, $x_H(t) \to 0$ exponentially and is called the transient solution $x_{tr}(t)$ (because it disappears as $t \to \infty$).

$x_P(t)$ as above is called the steady periodic solution $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit practical resonance. This can be an important phenomenon in electrical circuits, where amplifying signals is important. We don't generally want pure resonance or practical resonance in mechanical configurations.

3rd interesting phenomenon

We worked this Exercise on Friday...

Forced undamped oscillations: (We'll discuss forced damped oscillations on Monday next week.)
Exercise 1a) Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

1b) This superposition of two sinusoidal functions is periodic because there is a common multiple of their (shortest) periods. What is this (common) period?
1c) Compare your solution and reasoning with the display at the bottom of this page.

soltn from warmup

$$x(t) = -5\cos 5t + 5\cos 3t$$

$\cos \omega t$

ω = angular freq $\frac{\text{rad}}{\text{time}}$

$f = \frac{\omega}{2\pi}$ cycles/time

$T = \frac{2\pi}{\omega}$ time/cycle

period $T_1 = \frac{2\pi}{5}$

$\cos 5\left(\frac{2\pi}{5}\right) = \cos 2\pi = \cos 0$

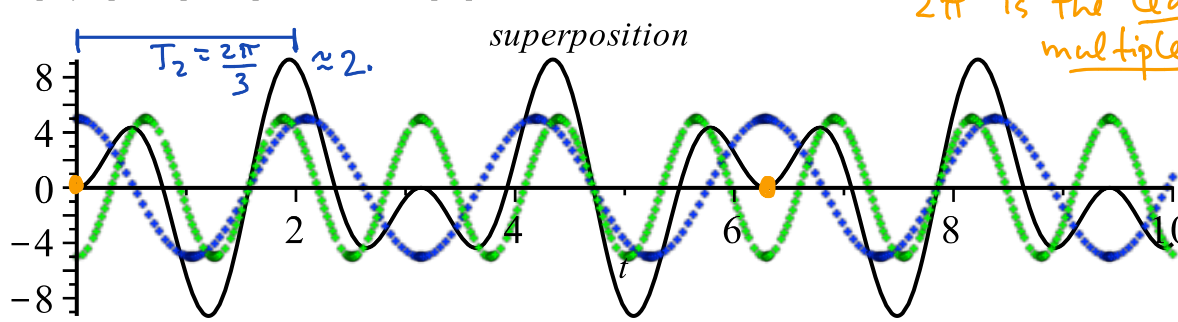
$T_2 = \frac{2\pi}{3}$

the period of the sum is (well it looks to be) 2π

2π is the least common multiple of T_1 & T_2

$2\pi = 5T_1$
$2\pi = 3T_2$

```
> with(plots) :
> plot1 := plot(-5·cos(5·t), t = 0..10, color = green, style = point) :
  plot2 := plot(5·cos(3·t), t = 0..10, color = blue, style = point) :
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0..10, color = black) :
  display({plot1, plot2, plot3}, title='superposition');
```



$T_1 = \frac{2\pi}{5} \approx 1.2$

$T = 2\pi$

return to this after finish Wed notes

In general: Use the method of undetermined coefficients to solve the initial value problem for $x(t)$, in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$:

no damping.

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

$$x''(t) + \omega_0^2 x(t) = \frac{F_0}{m}\cos\omega t$$

$$\big/\!\big/$$

$$L(x(t))$$

$$x_H(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$$

$$x_p(t) = d\cos\omega t$$

$$L(x_p(t)) = d\, L(\cos\omega t)$$
$$= d(\cos\omega t)\left[-\omega^2 + \omega_0^2\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega t$$

$$d(\omega_0^2 - \omega^2) = \frac{F_0}{m}$$

$$\Rightarrow d = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$x_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t$$

$$x = x_p + x_H$$

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t + c_1\cos\omega_0 t + c_2 \sin\omega_0 t$$

$$x(0) = x_0 = \frac{F_0}{m(\omega_0^2 - \omega^2)} + c_1 \Rightarrow c_1 = x_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$x'(0) = v_0 = c_2\omega_0 \Rightarrow c_2 = \frac{v_0}{\omega_0}$$

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t + \left[x_0 - \frac{F_0}{m(\omega_0^2 - \omega^2)}\right]\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

Solution:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\left(\cos(\omega_0 t) - \cos(\omega t)\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

There is an interesting beating phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$
$$-\left(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\right)$$
$$= 2\sin(\alpha)\sin(\beta) .$$

$$\alpha - \beta = \omega_0 t \;/\; \alpha + \beta = \omega t$$

Set $\alpha = \frac{1}{2}\left(\omega + \omega_0\right)t$, $\beta = \frac{1}{2}\left(\omega - \omega_0\right)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\left(2\sin\left(\frac{1}{2}(\omega + \omega_0)t\right)\sin\left(\frac{1}{2}(\omega - \omega_0)t\right)\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t) .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}\left(\omega - \omega_0\right), \qquad \text{period: } \frac{4\pi}{\left|\omega - \omega_0\right|} .$$

We will call half that period the beating period, as explained by the next exercise:

$$\text{beating period: } \frac{2\pi}{|\omega - \omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m|\omega^2 - \omega_0^2|}.$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)}\left(\cos(\omega_0 t) - \cos(\omega t)\right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)} 2 \sin\left(\frac{1}{2}(\omega + \omega_0)t\right)\sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t).$$

Exercise 2a) Use one of the formulas above to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\, t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

$\frac{F_0}{m} = 80$

$\omega_0 = 3$

$\omega = 3.1$

$x_0 = v_0 = 0$

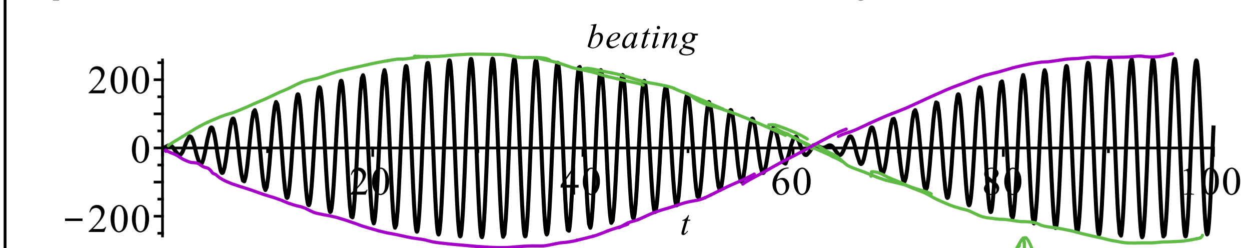
$\frac{\omega - \omega_0}{2} = .05$

$$x(t) = \underbrace{\frac{80}{(3.1)^2 - 3^2}}_{= 262.3} 2\sin(3.05t)\sin(.05t) + 0 + 0$$

period $\frac{2\pi}{.05} = 40\pi \approx 124$

beating period $= 20\pi$

> plot(262.3·sin(3.05·t)sin(.05·t), t = 0..100, color = black, title = `beating`);



oscillations between $\pm 262.3\sin(.05t)$ are caused by $\sin(3.05t)$

$x = 262.3\sin(.05t)$

Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned}
+\omega_0^2 \big(\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \big) \\
0 \big(\quad & x_p' = t\,(-A\omega_0 \sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \big) \\
+1 \big(\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \big)
\end{aligned}$$

$$L(x_p) = t(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\, t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

You can also get this solution by letting $\omega \to \omega_0$ in the beating formula. We will probably do it that way in class, on the next page.

in the case $\omega \neq \omega_0 = \sqrt{\frac{k}{m}}$ we copy the IVP solution in both forms, from previous page

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\cos(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)}\left(\cos\left(\omega_0 t\right) - \cos(\omega t)\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)$$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} 2\sin\left(\frac{1}{2}\left(\omega + \omega_0\right)t\right)\sin\left(\frac{1}{2}\left(\omega - \omega_0\right)t\right) + x_0\cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right).$$

If we let $\omega \to \omega_0$ this solution will converge to the resonance IVP solution on the previous page....

$$x(t) = \frac{F_0\ 2}{m(\omega+\omega_0)\cancel{(\omega-\omega_0)}} \sin\left(\frac{\omega+\omega_0}{2}t\right) \frac{\sin\left(\frac{\omega-\omega_0}{2}t\right)}{\left(\frac{\omega-\omega_0}{2}\right)t} \left(\frac{\cancel{\omega-\omega_0}}{2}t\right) \quad + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$\omega \to \omega_0$

$x_p(t) \to \frac{\cancel{2}F_0}{\cancel{2}m\omega_0}(\sin\omega_0 t)\frac{t}{2}$ on any interval $[0,M]$ $\qquad \to 1 \qquad \frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$

so in the limit,

$$x(t) = \frac{F_0}{2m\omega_0} t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

Resonance summary:

$$x''(t) + \omega_0^2\, x(t) = \frac{F_0}{m}\cos\left(\omega_0 t\right)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

has solution

$$x(t) = \frac{F_0}{2\, m \omega_0}\, t \sin\left(\omega_0\, t\right) + x_0 \cos\left(\omega_0 t\right) + \frac{v_0}{\omega_0}\sin\left(\omega_0 t\right)$$

<u>Exercise 3a)</u> Solve the IVP

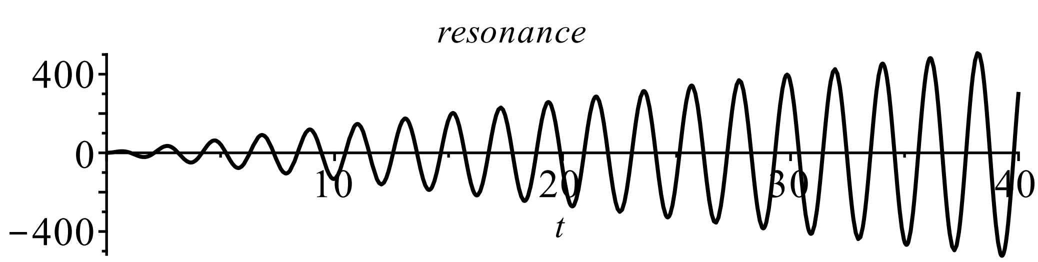
$$x'' + 9\, x = 80\cos(3\, t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Just use the general solution formula above this exercise and substitute in the appropriate values for the various terms.

$$x(t) = \frac{40}{3}\, t \sin 3t$$

<u>3b)</u> Compare the solution graph below with the beating graph in <u>exercise 2</u>.

```
> plot(40/3 t·sin(3· t), t = 0..40, color = black, title = `resonance`);
```




```
>
```

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)