Linear algebra to the rescue...just extend our previous discussion:
<u>Theorem</u>: Let

$$L : V \rightarrow W$$

be a linear transformation between vector spaces, where $dim\ V = dim\ W = n$. Then for each $f \in W$ there exists unique $y \in V$ solving

$$L(y) = f$$

if and only if the only solution $y \in V$ to $L(y) = 0$ is the trivial solution $y = 0$.

Proof: Let $\beta = \{y_1, y_2, ..., y_n\}$ be a basis for V. Let $C = \{f_1, f_2, ..., f_n\}$ be a basis for W. Let A be the matrix for L with respect to these two bases. Specfically, A is the matrix which converts β coordinates of input vectors into C coordinates of the outputs of L:

$$[L(y)]_C = A\ [y]_\beta$$

Then the equation $L(y) = f$ has a unique solution if and only if $[L(y)]_C = [f]_C$ does, i.e. if and only if the matrix equation

$$A\ [y]_\beta = [f]_C$$

has a unique solution. And this holds if and only if A reduces to the identity, i.e. if and only if $Nul\ A = \{0\}$, if and only if the only solution $y \in V$ to $L(y) = 0$ is the trivial solution $y = 0$.

QED

Method of undetermined coefficients ("Rule 2" page 190 text): Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

If L has a factor $(\mathrm{D} - r)^s$ and $e^{r x}$ is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s , as in Exercise 2. (There's also a current homework problem related to this case.) You may also need to use superposition, as in our Tuesday exercises, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	y_P	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1 + \ldots + b_m x^m$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)$	$r = 0$
$b_1 \cos(\omega x) + b_2 \sin(\omega x)$	$x^s\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$	$r = \pm i\omega$
$e^{a x}\left(b_1 \cos(\omega x) + b_2 \sin(\omega x)\right)$	$x^s e^{a x}\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$	$r = a \pm i\omega$
$b_0 e^{a x}$	$x^s c_0 e^{a x}$	$r = a$
$\left(b_0 + b_1 + \ldots + b_m x^m\right) e^{a x}$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right) e^{a x}$	$r = a$

$e^{0x} = 1$
$x e^{0x} = x$

Exercise 3) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for y_P. Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

3a) $y''' + 2y'' = x^2 + 6x$

(So the characteristic polynomial for $L(y) = 0$ is $r^3 + 2r^2 = r^2(r+2) = (r-0)^2(r+2)$.)

y_H: $y''' + 2y'' = 0$

$L = (D + 2I) \circ D^2$

$y_P = x^2(d_1 x^2 + d_2 x + d_3)$

case 1 guess

$V = \text{span}\{x^4, x^3, x^2\}$

$L : V \to W = \text{span}\{x^2, x^1, 1\}$

Differential equation solution:

$\cancel{y(x) = c_3 x^2 + c_2 x + c_1 + \frac{x^5}{180} + \frac{x^4}{12}}$

wrong DE

actual Wolfram alpha answer:

$y(x) = c_1 + c_2 x + c_3 e^{-2x} + \underbrace{\frac{x^4}{24} + \frac{5x^3}{12} - \frac{5x^2}{8}}_{y_P}$

3b) $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$

(So the characteristic polynomial for $L(y) = 0$ is

$r^2 - 4r + 13) = (r-2)^2 + 9 = (r-2+3i)^1(r-2-3i)^1$.)

$r = 2 \pm 3i$

$e^{(2+3i)x}$

$e^{(2-3i)x}$

$L = (D - (2+3i)I)^1 \circ (D - (2-3i)I)^1$

$y_P = x^1\left(d_1 e^{2x}\sin 3x + d_2 e^{2x}\cos 3x\right)$

Differential equation solution:

wrong DE

$\cancel{y(x) = c_1 e^{2x}\sin(3x) + c_2 e^{2x}\cos(3x) + \frac{4}{37}e^x\sin(3x) + \frac{24}{37}e^x\cos(3x)}$

actual Wolfram alpha answer

$y(x) = c_1 e^{2x}\sin 3x + c_2 e^{2x}\cos 3x$

$\underbrace{-\frac{2}{3}x e^{2x}\cos 3x}_{y_P}$

3c) $y'' + 5y' + 4y = 5\cos(2x) + 4e^{x} + 5e^{-x}$.
(So the characteristic polynomial for $L(y) = 0$ is $p(r) = r^2 + 5r + 4 = (r+4)(r+1)$.)

$r = -1, -4$

y_{P_1} for $L(y_{P_1}) = 5\cos 2x$

$y_{P_1} = d_1 \cos 2x + d_2 \sin 2x$

y_{P_2} for $L(y_{P_2}) = 4e^{x}$

$y_{P_2} = d_3 e^{x}$

y_{P_3} for $L(y_{P_3}) = 5e^{-x}$

$y_{P_3} = d_4 x e^{-x}$

$y = y_{P_1} + y_{P_2} + y_{P_3} + y_H$

$y_H = \text{span}\{e^{-x}, e^{-4x}\}$

$L = (D+4I)\circ(D+I)$

$(D+4I)\circ\underbrace{(D+I)\, xe^{-x}}_{e^{-x}}$ ...

Differential equation solution:

$$y(x) = \underbrace{c_1 e^{-4x} + c_2 e^{-x}}_{y_H} + \underbrace{\frac{5e^{-x}x}{3}}_{y_{P_3}} + \underbrace{\frac{2e^{x}}{5}}_{y_{P_2}} + \underbrace{\sin(x)\cos(x)}_{y_{P_1}}$$

$= \frac{1}{2}\sin 2x$

Math 2280-002
Fri Feb 22
3.6 forced oscillations

Announcements:
- Finish §3.5 (2^{nd} case of undetermined coef's).
- Start §3.6 (continue on Monday...)

in §3.6 we return to applications...

Warm-up Exercise: Solve the IVP for the forced undamped oscillator problem below, for $x(t)$

$$\begin{cases} x'' + 9x = 80\cos 5t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

a) x_H
b) x_p
c) IVP.

a) $x'' + 9x = 0$

$p(r) = r^2 + 9 = (r-3i)(r+3i)$

$$\boxed{x_H(t) = c_1\cos 3t + c_2 \sin 3t}$$

b) $x_p(t) = d_1 \cos 5t + d_2 \sin 5t$

$L(x_p) = d_1 L(\cos 5t) + d_2 L(\sin 5t)$

$= d_1\left(\underbrace{-25\cos 5t + 9\cos 5t}_{(-16\cos 5t)}\right) + d_2(-16\sin 5t) \overset{\text{want}}{=} 80\cos 5t + 0 \sin 5t$

$d_2 = 0$

$-16 d_1 = 80$

$d_1 = -5$

$$\boxed{x_p(t) = -5\cos 5t}$$

Actually, $L: \text{span}\{\cos 5t\} \to \text{span}\{\cos 5t\}$ because L only had even derivatives. So a guess of $x_p(t) = d_1 \cos 5t$ would've worked.

c) $x(t) = x_p + x_H$

$x(t) = -5\cos 5t + c_1\cos 3t + c_2 \sin 3t$

$x(0) = 0 = -5 + c_1 \Rightarrow c_1 = 5$

$x'(0) = 0 = 3c_2 \Rightarrow c_2 = 0$

$$\boxed{\text{sol'n} \quad x(t) = -5\cos 5t + 5\cos 3t}$$

$x'' + \omega_0^2 x = 0$

$\text{span}\{\cos\omega_0 t, \sin\omega_0 t\}$

Section 3.6: forced oscillations in mechanical systems (and as we shall see in section 3.7, also in electrical circuits) overview:
We study solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\, t)$$

using section 3.5 undetermined coefficients algorithms.

- undamped $(c = 0)$:
  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\, x = 0$$

  which has simple harmonic motion solutions

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C_0 \cos(\omega_0 t - \alpha)\ .$$

So for the non-homogeneous DE the section 5.5 method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\dfrac{k}{m}}$ $\Rightarrow x_P = A\cos(\omega\, t)$ because only even derivatives, we don't need

$\sin\,(\omega\, t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega\, t) + C_0 \cos(\omega_0 t - \alpha_0)\ .$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $A \approx C_0$ Beating!
- $\omega = \omega_0$ case 2 section 3.5 undetermined coefficients; since

$$p(r) = r^2 + \omega_0^2 = (r + i\omega_0)^1 (r - i\omega_0)^1$$

our undetermined coefficients guess is

$$x_P = t^1 \left(A\cos(\omega_0\, t) + B\sin(\omega_0\, t)\right)$$
$$\Rightarrow x = x_P + x_H = C\, t\cos(\omega\, t - \alpha) + C_0 \cos(\omega_0 t - \alpha_0)\ .$$

("pure" resonance!)

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega\, t) + B\sin(\omega\, t) = C\cos(\omega\, t - \alpha)$ (because the

roots of the characteristic polynomial are never purely imagninary $\pm\, i\,\omega$ when $c > 0$).

- underdamped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + e^{-p\,t} C_1 \cos(\omega_1 t - \alpha_1)\ .$
- critically-damped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + e^{-p\,t}(c_1 t + c_2)\ .$
- over-damped: $x = x_P + x_H = C\cos(\omega\, t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\ .$

- in all three damped cases on the previous page, $x_H(t) \to 0$ exponentially and is called the <u>transient solution</u> $x_{tr}(t)$ (because it disappears as $t \to \infty$).

$x_P(t)$ as above is called the <u>steady periodic solution</u> $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit <u>practical resonance.</u> This can be an important phenomenon in electrical circuits, where amplifying signals is important. We don't generally want pure resonance or practical resonance in mechanical configurations.

<u>Forced undamped oscillations</u>: (We'll discuss forced damped oscillations on Monday next week.)
<u>Exercise 1a)</u> Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

1b) This superposition of two sinusoidal functions <u>is</u> periodic because there is a common multiple of their (shortest) periods. What is this (common) period?
1c) Compare your solution and reasoning with the display at the bottom of this page.

soln from warmup

$$x(t) = -5\cos 5t + 5\cos 3t$$

period $T_1 = \frac{2\pi}{5}$

$\cos 5\left(\frac{2\pi}{5}\right) = \cos 2\pi = \cos 0$

$T_2 = \frac{2\pi}{3}$

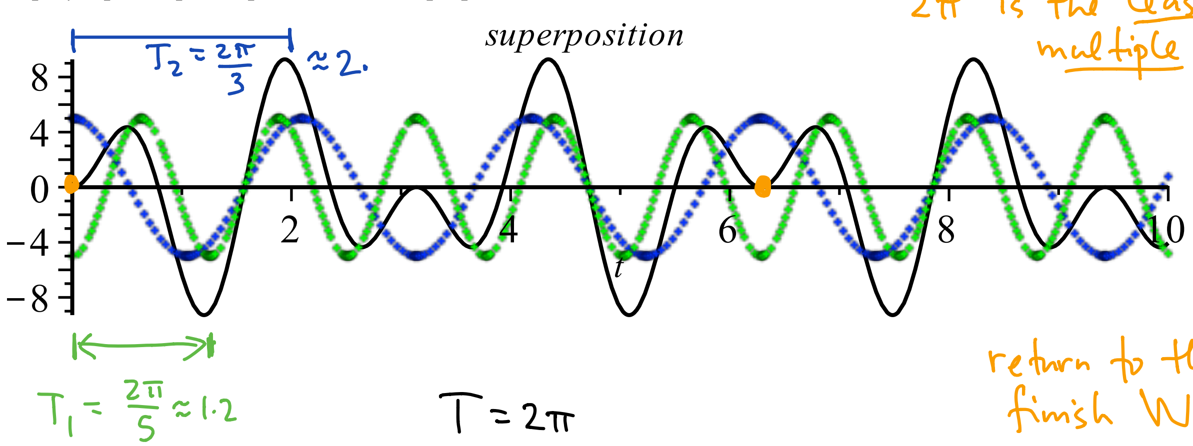
$\cos \omega t$

ω = angular freq $\frac{\text{rad}}{\text{time}}$

$f = \frac{\omega}{2\pi}$ cycles/time

$T = \frac{2\pi}{\omega}$ time/cycle

```
> with(plots) :
> plot1 := plot(-5·cos(5·t), t = 0 ..10, color = green, style = point) :
  plot2 := plot(5·cos(3·t), t = 0 ..10, color = blue, style = point) :
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0 ..10, color = black) :
  display({plot1, plot2, plot3}, title='superposition');
```



$T = 2\pi$

the period of the sum is (well it looks to be) 2π
2π is the <u>least common multiple</u> of T_1 & T_2

$2\pi = 5T_1$
$2\pi = 3T_2$

return to this after finish Wed notes