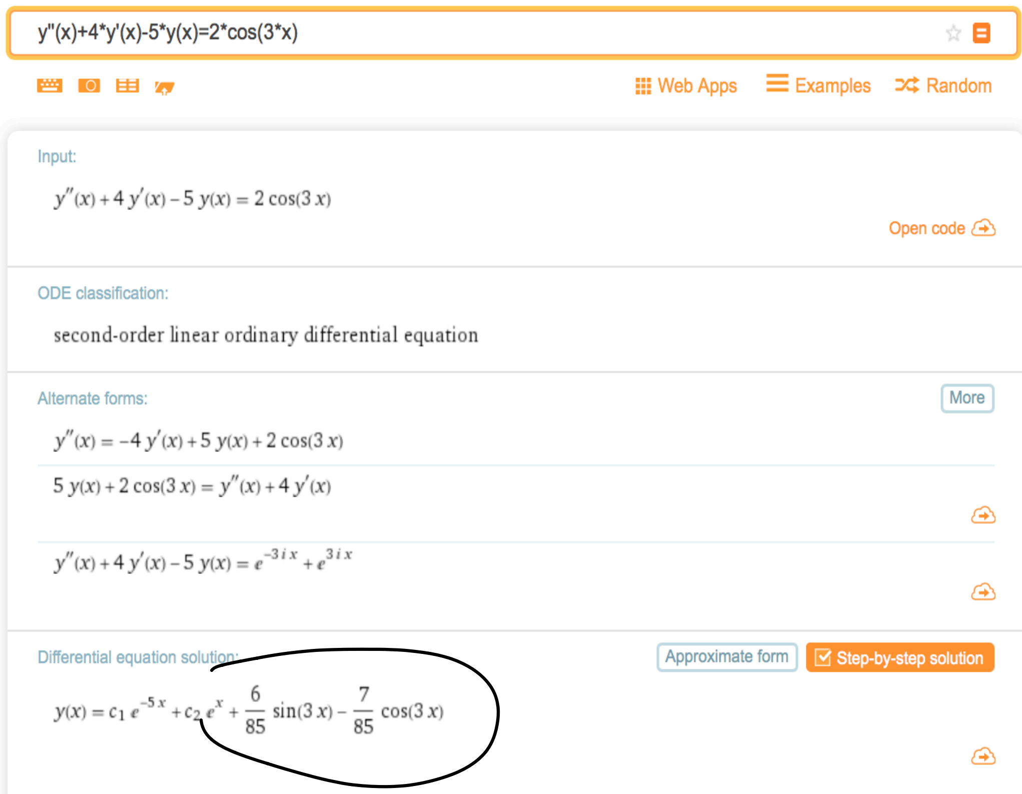


All of the previous exercises rely on:

<u>Method of undetermined coefficients</u> (<u>base case</u>): Let $L : V \to V$ be a linear transformation, with V a finite dimensional vector space, and let $f \in V$. Then $\exists!\ y_P \in V$ with $L(y_P) = f$ if and only if the only $y \in V$ for which $L(y) = 0$ is $y = 0$.

<u>why:</u> You definitely learned this fact in Math 2270, for the special case of matrix transformations $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\underline{x}) = A_{n \times n}\underline{x}$. (Each non-homogeneous matrix equation $A\,\underline{x} = \underline{b}$ has a unique solution $\underline{x}$ if and only if A reduces to the identity matrix I, if and only if the only solution to the homogeneous equation $\underline{A}\,\underline{x} = \underline{0}$ is $\underline{x} = \underline{0}$.) The theorem above ~~is a generalization~~ of this fact to general linear transformations $L : V \to V$. In fact, if we pick a basis $\{\underline{v}_1, \underline{v}_2, \ldots\ \underline{v}_n\}$ for V, this base case of undetermined coefficients follows from the fact that the (square) ~~matrix A for L~~ with respect to this basis is invertible if and only if $Nul\,A = \{\underline{0}\}$.

$\Downarrow$ "β" basis

$L y_p = f$ has unique soln in V?

$\iff [L(y_p)]_\beta = [f]_\beta$ (iff coords agree)

$A\,[y_p]_\beta = [f]_\beta$ has unique soln.

Wed Feb 20:
3.5: Finding y_P for non-homogeneous linear differential equations, continued,

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems).

Announcements:

- Pick up your exams
- Quiz today on yesterday's & today's material

Warm-up Exercise: Pick the appropriate form for undetermined coefficients $y_P(x)$ "guess"

a) $y'' + 6y' + 9y = 4e^{-2x}$

$y_P = de^{-2x}$

$V = \text{span}\{e^{-2x}\}$ 1-dim'l space

$L: V \to V$

$L(de^{-2x}) = d\,L(e^{-2x})$
$= d\,e^{-2x}[\underbrace{4 - 12 + 9}_{1}]$
$= de^{-2x} \overset{\text{want}}{=} 4e^{-2x}$ pick $d = 4$

b) $y'' + 6y' + 9y = 18\cos 3x$

$y_P = d_1 \cos 3x + d_2 \sin 3x$

$V = \text{span}\{\cos 3x, \sin 3x\}$
2-dim'l

on exam
$y_P(x) = \sin 3x$

c) $y'' + 6y' + 9y = 2e^{-3x}$

naive guess $y_P = de^{-3x}$

$L(y_P) = d\,L(e^{-3x}) = d\,e^{-3x}(9 - 18 + 9) = 0$

oh oh.

Note: y_H:

$y'' + 6y' + 9y = 0$

$p(r) = r^2 + 6r + 9 = (r+3)^2 = 0 \quad r = -3$

$y_H(x) = c_1 e^{-3x} + c_2 x e^{-3x}$

return to this in class ... $y_P(x) = d x^2 e^{-3x}$

$L = D^2 + 6D + 9I$
$= (D + 3I) \circ (D + 3I)$

$y_P = x^2 e^{-3x}$

$L(dx^2 e^{-3x}) = d\,L(x^2 e^{-3x}) = d\,(D+3I) \circ \underbrace{(D+3I)\,x^2 e^{-3x}}_{2xe^{-3x}}$

let $d = 1$

$\underbrace{}_{2e^{-3x}}$

$= 2de^{-3x} \overset{\text{want}}{=} 2e^{-3x}$

On Tuesday we discussed the base case of undetermined coefficients:

Method of undetermined coefficients (base case): If you wish to find a particular solution y_P, i.e. $L(y_P) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that
(i) $L : V \to V$, i.e. L transforms functions in V into functions which are also in V; and
(ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_P \in V$ with $L(y_P) = f$.

Exercise 1) Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

(*assuming* the only such solution in your specified subspace that would solve the homogeneous DE is the zero solution):

1a) $L(y) = x^3 + 6x - 5$

$$y_p = d_1 x^3 + d_2 x^2 + d_3 x + d_4$$

$$V = \text{span}\{x^3, x^2, x, 1\} \qquad L : V \to V$$

1b) $L(y) = 4e^{2x}\sin(3x)$

$$y_p = d_1 e^{2x} \sin 3x + d_2 e^{2x} \cos 3x$$

$$V = \text{span}\{e^{2x}\sin 3x, e^{2x}\cos 3x\}$$

1c) $L(y) = x\cos(2x)$

$$y_p = d_1 x\cos 2x + d_2 x\sin 2x + d_3 \cos 2x + d_4 \sin 2x$$

$$V = \text{span}\{x\cos 2x, x\sin 2x, \cos 2x, \sin 2x\}$$

$$L : V \to V$$

$$(D - rI)\, f(x) e^{rx} = f'(x) e^{rx}$$
$$= f'(x) e^{rx} + \cancel{r f(x) e^{rx}} - \cancel{r f(x) e^{rx}}$$

y_H: $y'' + 4y' - 5y = 0$

$p(r) = r^2 + 4r - 5$

$= (r+5)(r-1)$

BUT LOOK OUT

<u>Exercise 2a)</u> Find a particular solution to

$$L(y) = y'' + 4\,y' - 5\,y = 4\,e^x\,.$$

Hint: since $\boxed{y_H = c_1 e^x + c_2 e^{-5\,x}}$, a guess of $y_P = a\,e^x$ will <u>not</u> work (and *span*$\{e^x\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_P = d\,x\,e^x$$

and factor $L = D^2 + 4\,D - 5I = [D + 5I] \circ [D - I]$.

Case 1 does not apply

naive:

$$-5\,(y_p = d x e^x)$$
$$+4\,(y_p' = d e^x + d x e^x)$$
$$+1\,(y_p'' = 2 d e^x + d x e^x)$$

$$L(y_p) = x e^x \underbrace{(-5d + 4d + d)}_{0}$$
$$+ e^x (4d + 2d)$$
$$= 6 d e^x \overset{\text{want}}{=} 4 e^x$$

$d = 2/3$ $\quad \boxed{y_p = \frac{2}{3} x e^x}$

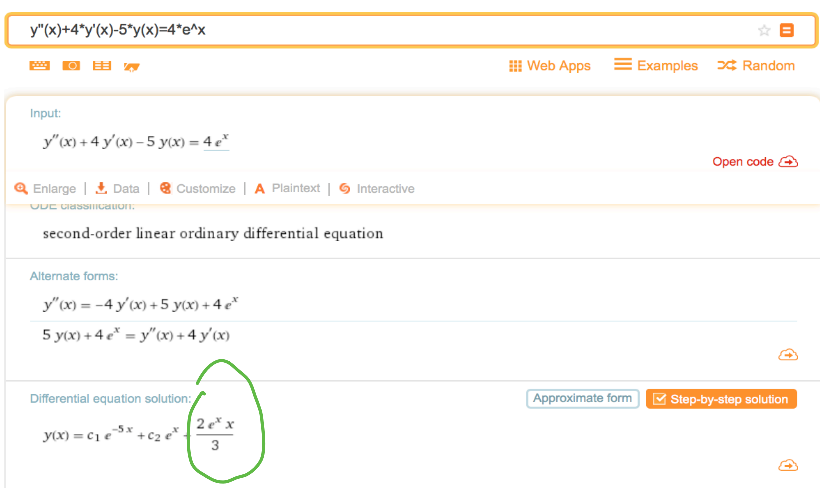
Smart way "$f(x)$"

$$(D + 5I) \circ \underbrace{(D - I)(d\,x\,e^x)}$$

$$(D + 5I)\, d e^x$$
$$= d e^x + 5 d e^x$$
$$= 6 d e^x$$

Same!

<u>2b)</u> check work with technology



Linear algebra to the rescue...just extend our previous discussion:
<u>Theorem</u>: Let

$$L : V \to W$$

be a linear transformation between vector spaces, where $dim\ V = dim\ W = n$. Then for each $f \in W$ there exists unique $y \in V$ solving

$$L(y) = f$$

if and only if the only solution $y \in V$ to $L(y) = 0$ is the trivial solution $y = 0$.

Proof: Let $\beta = \{y_1, y_2, ..., y_n\}$ be a basis for V. Let $C = \{f_1, f_2, ..., f_n\}$ be a basis for W. Let A be the matrix for L with respect to these two bases. Specfically, A is the matrix which converts β coordinates of input vectors into C coordinates of the outputs of L:

$$[L(y)]_C = A\ [y]_\beta$$

Then the equation $L(y) = f$ has a unique solution if and only if $[L(y)]_C = [f]_C$ does, i.e. if and only if the matrix equation

$$A\ [y]_\beta = [f]_C$$

has a unique solution. And this holds if and only if A reduces to the identity, i.e. if and only if $Nul\ A = \{0\}$, if and only if the only solution $y \in V$ to $L(y) = 0$ is the trivial solution $y = 0$.

QED

<u>Method of undetermined coefficients ("Rule 2" page 190 text)</u>: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

If L has a factor $(D - r)^s$ and e^{rx} is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s, as in <u>Exercise 2</u>. (There's also a current homework problem related to this case.) You may also need to use superposition, as in our Tuesday exercises, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	y_P	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1 + \ldots + b_m x^m$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)$	$r = 0$
$b_1 \cos(\omega x) + b_2 \sin(\omega x)$	$x^s\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$	$r = \pm i\omega$
$e^{ax}\left(b_1 \cos(\omega x) + b_2 \sin(\omega x)\right)$	$x^s e^{ax}\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$	$r = a \pm i\omega$
$b_0 e^{ax}$	$x^s c_0 e^{ax}$	$r = a$
$\left(b_0 + b_1 + \ldots + b_m x^m\right)e^{ax}$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)e^{ax}$	$r = a$