Math 2280-002
Week 7: 3.5-3.6
Tues Feb 19
3.5: Finding particular solutions to nonhomogeneous constant coefficient linear differential equations.

Announcements:

- I've only just started grading exams — should return them tomorrow (solutions are posted on CANVAS)
- Quiz tomorrow on §3.5 material
- HW is due next week.

on HW, you can already do 3.4 5 6 23, w7.3
after today w7.1 abc

Warm-up Exercise: Let $L(y) = y'' - 4y' - 5y$.

Let $V = \text{span}\{1, x\}$

$y_1(x) = 1$ $\quad y_2(x) = x$.

2270? Find the matrix for L with respect to $\beta = \{1, x\}$.

Hint: If you don't remember how to do that, compute $L(d_1 + d_2 x)$ $\quad (d_1, d_2$ constants$)$

$$
\begin{aligned}
-5\ &(y(x) = d_1 + d_2 x)\\
-4\ &(y'(x) = d_2)\\
+1\ &(y''(x) = 0)
\end{aligned}
$$

$$L(y) = -5d_1 - 4d_2 - 5d_2 x$$
$$= c_1 + c_2 x$$

$$\underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{[L(y)]_\beta} = \underbrace{\begin{bmatrix} -5 & -4 \\ 0 & -5 \end{bmatrix}}_{[L]_\beta} \underbrace{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}}_{[y(x)]_\beta}$$

$[L]_\beta$: matrix of L with respect to basis β

$[y(x)]_\beta$: coords of $y(x)$ with respect to β

transforms input coords ("weights") into output coords

quick way to compute $[L]_\beta$

$$= \begin{bmatrix} [L(y_1)]_\beta & [L(y_2)]_\beta \end{bmatrix}$$

$$= \begin{bmatrix} [L(\text{"1"})]_\beta & [L(\text{"x"})]_\beta \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -4 \\ 0 & -5 \end{bmatrix}$$

$L(1) = -5$; $L(\text{"x"}) = -4 - 5x$

Section 3.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems, and because sometimes a good choice for y_P contains the most essential information in dynamical systems problems).

• The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 3.5.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise as we go through today's notes - *at its core this method is based on the circle of ideas related to the matrix of a linear transformation.*

Exercise 1a) Find a particular solution $y_P(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 3 + 10x.$$

Hint: try $y_P(x) = d_1 + d_2 x$ because L transforms such functions into ones of the same form $b_1 + b_2 x$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 3, b_2 = 10$.

try $-5(y_P = d_1 + d_2 x)$

$+4(y_P' = d_2)$

$+1(y_P'' = 0)$

$L(y_P) = (-5d_1 + 4d_2) - 5d_2 x \overset{\text{want}}{=} 3 + 10x$

"1" coeffs: $-5d_1 + 4d_2 = 3$

"x" coeffs: $-5d_2 = 10$

$\Rightarrow d_2 = -2$

$-5d_1 - 8 = 3$

$-5d_1 = 11$

$d_1 = -\frac{11}{5}$

$$\boxed{y_P(x) = -\frac{11}{5} - 2x}$$

Exercise 1b) Linear algebra interpretation of previous page: Let $\beta = \{y_1(x) = 1, y_2(x) = x\}$ be a basis for the two-dimensional vector space $P_1 = \{y(x) = d_1 + d_2 x, d_1, d_2 \in \mathbb{R}\}$ of polynomials of degree less than or equal to one. Note that our

$$L(y) := y'' + 4y' - 5y$$

transforms P_1 back to itself, $L : P_1 \to P_1$. Use the matrix for L with respect to the basis β to re-find the particular solution $y_P \in P_1$ to

$$L(y) := y'' + 4y' - 5y = \underline{3 + 10x}.$$

almost from warmup. (Sorry!)

If $y(x) = d_1 + d_2 x$

$L(y(x)) = c_1 + c_2 x$

then

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$\uparrow [L(1)]_\beta$ $\quad \uparrow [L(\text{"}x\text{"})]_\beta$

$L(x) = \text{"}x\text{"}'' + 4\text{"}x\text{"}' - 5x$

$= 4 - 5x$

$[L(x)]_\beta = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

"same" as previous page"

$$\begin{bmatrix} 3 \\ 10 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} -5 & -4 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 10 \end{bmatrix} = \begin{bmatrix} -\frac{55}{25} \\ -\frac{50}{25} \end{bmatrix} = \begin{bmatrix} -\frac{11}{5} \\ -2 \end{bmatrix}$$

$y_P(x) = -\frac{11}{5} - 2x$ ✓

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4\,y' - 5\,y = 10\,x + 3$$

$y = y_P + y_H$

$\downarrow$

$y'' + 4y' - 5y = 0$

$p(r) = r^2 + 4r - 5$

$= (r+5)(r-1)$

roots $r = -5, 1$ $\quad y_H(x) = c_1 e^{-5x} + c_2 e^{x}$

$y = y_p + y_H$

$$\boxed{y = -\frac{11}{5} - 2x + c_1 e^{-5x} + c_2 e^{x}}$$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4\,y' - 5\,y = 14\,e^{2\,x}.$$

Hint: try $y_P = d\,e^{2\,x}$ because L transforms functions of that form into ones of the form $b\,e^{2\,x}$, i.e. $L\left(d\,e^{2\,x}\right) = b\,e^{2\,x}$. "$d$" is your "undetermined coefficient" for $b = 14$. (In terms of linear algebra, we are using the fact that for the one-dimensional vector space $V = span\left\{e^{2\,x}\right\}$, $L(V) = V$.

$-5\,(y_p = d e^{2x})$

$+4\,(y_p' = 2d e^{2x})$

$+1\,(y_p'' = 4d e^{2x})$

$L(y_p) = d e^{2x}(-5+8+4) = 7 d e^{2x} = 14 e^{2x} \Rightarrow d = 2$

$$\boxed{y_p(x) = 2e^{2x}}$$

Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6\,. \quad = 14e^{2x} - 2\,\underbrace{(3+10x)}$$

$\uparrow L(y_{P_2})$, $\quad y_{P_2} = 2e^{2x}$

$\uparrow L(y_{P_1})$, $\quad y_{P_1} = -\frac{11}{5} - 2x$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6$$
$$y(0) = 4$$
$$y'(0) = -4\,.$$

$y = y_{P_2} - 2y_{P_1} + y_H$

$y(x) = 2e^{2x} - 2\left(-\frac{11}{5} - 2x\right) + c_1 e^{-5x} + c_2 e^{x}$

$$\boxed{y(x) = 2e^{2x} + \frac{22}{5} + 4x + c_1 e^{-5x} + c_2 e^{x}}$$

4c) Check your answer with technology.

y"(x)+4*y'(x)-5*y(x)=14*exp(2*x)-20*x-6, y(0)=4, y'(0)=-4

Web Apps Examples Random

Input:

$$\{y''(x) + 4\,y'(x) - 5\,y(x) = 14\exp(2\,x) - 20\,x - 6,\ y(0) = 4,\ y'(0) = -4\}$$

Open code

ODE classification:

second-order linear ordinary differential equation

Alternate forms:

More

$$\{y''(x) = -4\,y'(x) + 5\,y(x) - 20\,x + 14\,e^{2\,x} - 6,\ y(0) = 4,\ y'(0) = -4\}$$

$$\{5\,y(x) + 14\,e^{2\,x} = y''(x) + 4\,y'(x) + 20\,x + 6,\ y(0) = 4,\ y'(0) = -4\}$$

$$\{y''(x) + 4\,y'(x) - 5\,y(x) = 2\,(-10\,x + 7\,e^{2\,x} - 3),\ y(0) = 4,\ y'(0) = -4\}$$

Differential equation solution:

Approximate form Step-by-step solution

$$y(x) = 4\,x + \frac{8\,e^{-5\,x}}{5} - 4\,e^{x} + 2\,e^{2\,x} + \frac{22}{5}$$

y_p

$c_1 = 8/5$

$c_2 = -4$

Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = 2\cos(3x).$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \to V$. If L is an invertible linear transformation then there will be exactly one particular solution y_P in V for $L(y_P) = f$.

- In Exercise 1 $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2 x$.
- In Exercise 3 $V = span\{e^{2x}\}$ and so we guessed $y_P = d\, e^{2x}$.

- What's the smallest subspace V ~~we can take in the current exercise~~? Can you see why $V = span\{\cos(3x)\}$ and a guess of $y_P = d\cos(3x)$ won't work?

operator will create mults of sin 3x as well

$$V = span\{\cos 3x, \sin 3x\}$$

$$\begin{aligned}
-5\,(\ y_P &= d_1\cos 3x + d_2 \sin 3x\ )\\
+4\,(\ y_P' &= -3d_1\sin 3x + 3d_2\cos 3x\ )\\
+1\,(\ y_P'' &= -9d_1\cos 3x - 9d_2\sin 3x\ )
\end{aligned}$$

$$L(y_P) = \cos 3x\,(-5d_1 + 12d_2 - 9d_1) + \sin 3x\,(-5d_2 - 12d_1 - 9d_2) \overset{\text{want}}{=} \cos 3x\,(2) + \sin 3x\,(0)$$

$$\begin{aligned}
-14d_1 + 12d_2 &= 2\\
-12d_1 - 14d_2 &= 0
\end{aligned}$$

$$\underbrace{\begin{bmatrix} -14 & 12 \\ -12 & -14 \end{bmatrix}}_{[L]_\beta}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

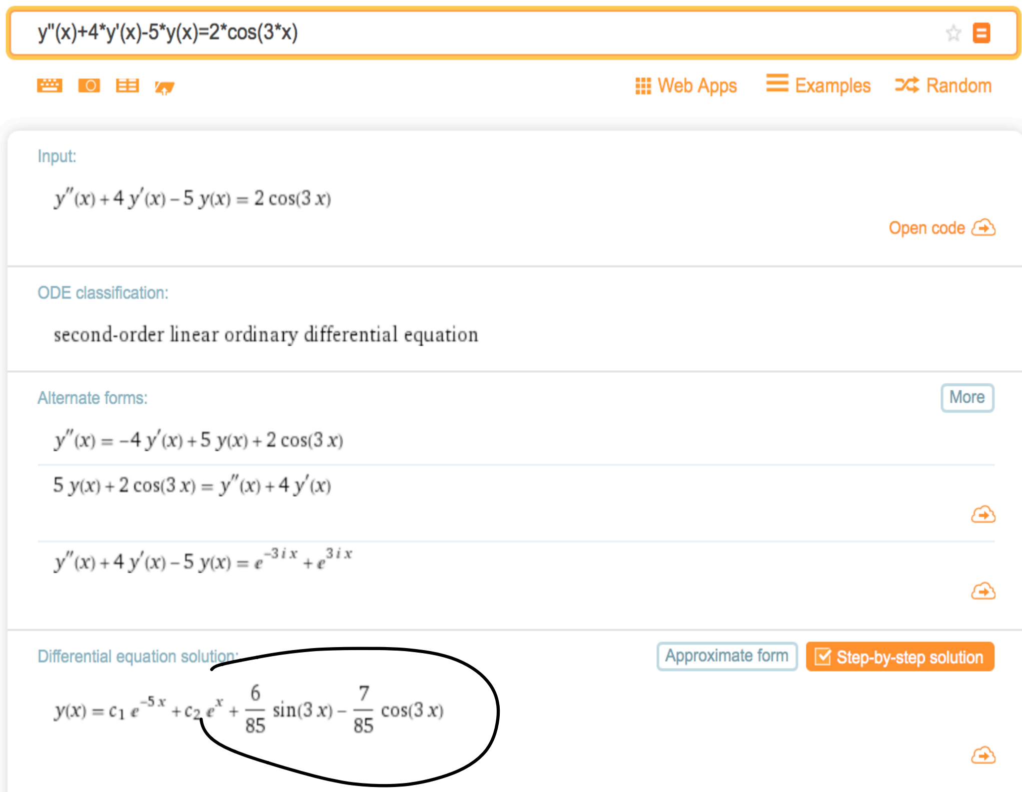
$$\beta = \{\cos 3x, \sin 3x\}.$$

$\div 2$: $$\begin{bmatrix} -7 & 6 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{y_P = \frac{-7}{85}\cos 3x + \frac{6}{85}\sin 3x}$$

$$\begin{array}{r} 49 \\ \underline{36} \\ 85 \end{array}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{85}\begin{bmatrix} -7 & -6 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7/85 \\ 6/85 \end{bmatrix}$$



All of the previous exercises rely on:

<u>Method of undetermined coefficients</u> (<u>base case</u>): Let $L : V \to V$ be a linear transformation, with V a finite dimensional vector space, and let $f \in V$. Then $\exists!\ y_P \in V$ with $L(y_P) = f$ if and only if the only $y \in V$ for which $L(y) = 0$ is $y = 0$.

<u>why:</u> You definitely learned this fact in Math 2270, for the special case of matrix transformations $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\underline{\boldsymbol{x}}) = A_{n \times n}\underline{\boldsymbol{x}}$. (Each non-homogeneous matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution $\underline{\boldsymbol{x}}$ if and only if A reduces to the identity matrix I, if and only if the only solution to the homogeneous equation $\underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ is $\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$.) The theorem above is a generalization of this fact to general linear transformations $L : V \to V$. In fact, if we pick a basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots.\ \underline{\boldsymbol{v}}_n\}$ for V, this base case of undetermined coefficients follows from the fact that the (square) matrix A for L with respect to this basis is invertible if and only if $Nul\,A = \{\underline{\boldsymbol{0}}\}$.