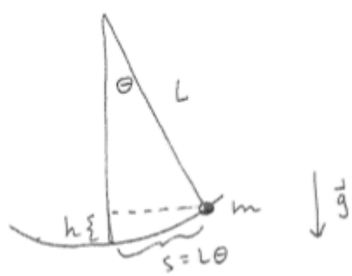
Wed Feb 13:
3.4 experiments, and review for exam

Announcements: review tomorrow (go over practice exam)
1:00 – 2:20 LCB 323 upstairs.

Warm-up Exercise: We'll be taking measurements for experiments

Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of energy, as indicated below. Conservation of energy is an important tool in deriving differential equations, in a number of different contexts. Today we will test both the pendulum model and the mass-spring model with actual experiments (in the undamped cases), to see if the predicted periods $T = \frac{2\pi}{\omega_0}$ correspond to experimental reality.

1) pendulum



$\theta = \theta(t)$

conservative system $KE + PE = \text{const.}$

$$\frac{1}{2} m v^2 + mgh = \text{const}$$

$$s = L\theta$$

$$v = \frac{ds}{dt} = L\theta'(t) \qquad h = L - L\cos\theta = L(1-\cos\theta)$$

so,

$$\frac{1}{2} m L^2 (\theta'(t))^2 + mgL(1-\cos(\theta(t))) \equiv \text{const}$$

$$D_t: \quad mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$$

$$\underbrace{mL\theta'}_{\substack{\neq 0 \text{ except} \\ \text{at isolated} \\ \text{times}}}(L\theta'' + g\sin\theta) \equiv 0$$

$\rightsquigarrow$ deduce eqtn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

linearize

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

$$\omega_0 = \sqrt{\frac{g}{L}}$$

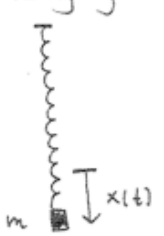
$$\theta(t) = C\cos(\omega_0 t - \alpha)$$

non-linear DE

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$

$\sin\theta \approx \theta$ θ small is excellent approx (alternating series test)

2) hanging mass-spring:



$$mx'' = -kx$$

$$mx'' + kx = 0$$

$$\boxed{x'' + \frac{k}{m}x = 0}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (These are the actual numbers from our class measurement).

> *restart* :
> *Digits* := 4 :

> $L := 1.531;$
> $g := 9.806;$
> $\omega := \sqrt{\dfrac{g}{L}}$; # *radians per second*
> $f := \mathit{evalf}(\omega/(2\cdot \mathrm{Pi}))$; # *cycles per second*
> $T := 1/f$; # *seconds per cycle*

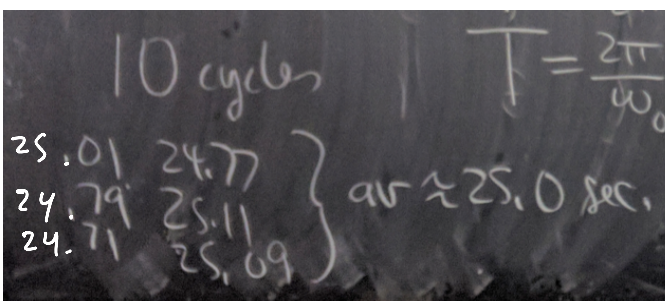
$$L := 1.531$$
$$g := 9.806$$
$$\omega := 2.530803050$$
$$f := 0.4027898154$$
$$T := 2.482684422 \qquad \textbf{(1)}$$

Experiment: The various measured times for the same 10 cycles and their average are recorded below:

> $\dfrac{(25.01 + 24.79 + 24.71 + 24.77 + 25.11 + 25.09)}{6}$;

$$24.91333333 \qquad \textbf{(2)}$$

Experimental period: 2.49 ± .02 seconds. Model prediction: 2.48 seconds. Wow!



Mass-spring:

compute Hooke's constant: actual numbers from class measurements

> $84.6 - 68.8;$ *#displacement from extra 50g*

$$15.8 \tag{3}$$

> $k := \frac{.05 \cdot 9.806}{.158};$ *# solve k·x=m·g for k.*

$$k := 3.103164557 \tag{4}$$

> $m := .1;$ *# mass for experiment is 100g*

$\omega := \sqrt{\frac{k}{m}};$ *# predicted angular frequency*

$f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right);$ *# predicted frequency*

$T := \frac{1}{f};$ *# predicted period*

$$m := 0.1$$

$$\omega := 5.570605494$$

$$f := 0.8865894005$$

$$T := 1.127917838 \tag{5}$$

Experiment: The actual measurements and their average are shown below, along with the measured period

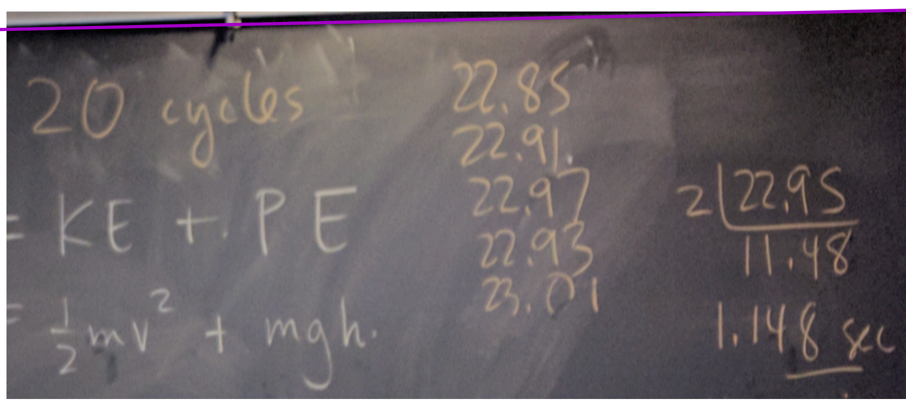
> $\frac{(22.85 + 22.91 + 22.97 + 22.93 + 23.01)}{5};$

$$22.93400000 \tag{6}$$

> $\frac{22.93400000}{20};$

$$1.146700000 \tag{7}$$

Experimental period: 1.147 ± ~~.05~~ .005 seconds. Model prediction: 1.128 seconds. Close, but something's not quite right....



We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

<u>Improved mass-spring model</u>
Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work}.$$

$$PE_{work} = \int_0^x k\,s\,\mathrm{d}s = \frac{1}{2}k\,x^2, \quad KE_{mass} = \frac{1}{2}m\,(x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu\,x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ , as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2}\left(\mu\,x'(t)\right)^2\left(m_{spring}\,\mathrm{d}\mu\right)$$

$$= \frac{1}{2}m_{spring}(x'(t))^2\int_0^1 \mu^2\,d\,\mu = \frac{1}{6}m_{spring}(x'(t))^2 \;.$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3}m_{spring}\right)(x'(t))^2 + \frac{1}{2}k\,x^2 = \frac{1}{2}M(x'(t))^2 + \frac{1}{2}k\,x^2,$$

where

$$M = m + \frac{1}{3}m_{spring}$$

$\mathrm{D}_t(TE) = 0 \Rightarrow$

$$M\,x'(t)\,x''(t) + k\,x(t)\,x'(t) = 0\,.$$
$$x'(t)\,(M\,x'' + k\,x) = 0\,.$$

Since $x'(t) = 0$ only at isolated $t-$values, we deduce that the corrected equation of motion is

$$(M\,x'' + k\,x) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{m + \frac{1}{3}m_{spring}}}\;.$$

Does this lead to a better comparison between model and experiment?

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> ms := .0103;  # spring has mass 10.3 g
  M := m + 1/3·ms;  # "effective mass"
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$$ms := 0.0103$$

$$M := 0.1034333333 \tag{8}$$

\> $\omega := \sqrt{\frac{k}{M}}$; *# predicted angular frequency*

$f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right)$; *# predicted frequency*

$T := \frac{1}{f}$; *# predicted period*

$$\omega := 5.477370807$$

$$f := 0.8717506390$$

$$T := 1.147117026 \tag{9}$$

\>

With the improved model, we have:

Experimental period: 1.147 $\pm$.~~05~~ seconds. Model prediction: 1.147 seconds. Freakishly close.

.005

Exam 1 is this Friday February 15, from 12:50-1:50 p.m.

start 5 minutes early
end 5 minutes late

This exam will cover textbook material from 1.1-1.5, 2.1-2.4, 3.1-3.4. The exam is closed book and closed note. You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed.

2.4 Euler
not 2.5 or 2.6 (no improved Euler or R.K.)

I recommend trying to study by organizing the conceptual and computational framework of the course so far. Only then, test yourself by making sure you can explain the concepts and do typical problems which illustrate them. The class notes and text should have explanations for the concepts, along with worked examples. Old homework assignments and quizzes are also a good source of problems.

I will have posted one or two practice exams and solutions, from recent times I've I taught Math 2280. Theyshould give you a feel for how I structure exams and address course topics. I'll go over a practice exam on Thursday February 14, 1:00-2:20, ~~location TBA~~.

LCB 323

My goal is to test key ideas/computations
without making overly difficult problems

Exam 1 Review Questions

1a) What is a differential equation? What is its order? What is an initial value problem, for a first or second order (or higher order) DE?

an equation involving a function "y(x)", and some of its derivatives

highest deriv that appears

also specify initial conditions which you want sol's to satisfy

n^{th} order
n I.C.'s

1b) How do you check whether a function solves a differential equation? An initial value problem?

Subs $y(x)$ into DE
do you get a true identity

does $y(x)$ also satisfy the IC's

1c) What is the connection between a first order differential equation and a slope field for that differential equation? The connection between an IVP and the slope field?

If $y' = f(x,y)$
$y(x_0) = y_0$ $\longrightarrow$ start @ (x_0, y_0) the solution graph stays tangent to slope field.

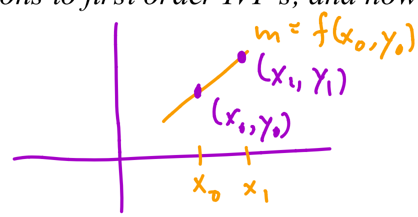
1d) Do you expect solutions to IVP's to exist, at least for values of the input variable close to its initial value? Why? Do you expect uniqueness? What does the existence-uniqueness theorem say? What can cause solutions to not exist beyond a certain input variable value?

$y' = f(x,y)$ If $f(x,y)$ is cont near (x_0, y_0) solutns exist

if $\frac{\partial f}{\partial y}$ also cont near (x_0, y_0), the soltn to IVP is unique (near $x = x_0$)

1e) What is Euler's numerical method for approximating solutions to first order IVP's, and how does it relate to slope fields?

(x_0, y_0)
$x_1 = x_0 + h.$
$y_1 = y_0 + h f(x_0, y_0)$



1f) Can you recognize the first order differential equations for which we've studied solution algorithms, even if the DE is not automatically given to you pre-set up for that algorithm? Do you know the algorithms for solving these particular first order DE's?

1^{st} order linear $y'(x) + P(x)y = Q(x)$

1^{st} order separable $\frac{dy}{dx} = f(x)\, g(y)$ ..

$y(x)$ $y' = f(y)$ constant sol'n $y(x) \equiv c$

i.e. $f(c) = 0$

2a) What's an autonomous differential equation? What's an equilibrium solution to an autonomous differential equation? What is a phase diagram for an autonomous first order DE, and how do you construct one? How does a phase diagram help you understand stability questions for equilibria? What does the phase diagram for an autonomous first order DE have to do with the slope field? What models did we study, related to population dynamics?

phase diagram is like a compression of the (t–x) slope field info onto the x-axis

If we write DE for $x(t)$ as $x' = f(x)$ then phase diagram is a number line containing the equilibrium solutions and arrows indicating whether $x(t)$ is increasing or decreasing on the intervals between the equil. solutions (you check the sign of $x'(t) = f(x)$ on the subintervals).

phase diagram lets you deduce stability:

$\rightarrow \underset{c}{\bullet} \leftarrow$ asymptotically stable

$\rightarrow \underset{c}{\bullet} \rightarrow$ or $\leftarrow \underset{c}{\bullet} \leftarrow$ unstable (semi-stable)

$\leftarrow \bullet \rightarrow$ unstable

2b) Can you convert a description of a dynamical system in terms of rates of change, or a geometric configuration in terms of slopes, into a differential equation? What are the models we've studied carefully in Chapters 1-2? What sorts of DE's and IVP's arise? Can you solve these basic application DE's, once you've set up the model as a differential equation and/or IVP?

models
- exponential growth & decay – separable & linear
- Newton's law of cooling – separable & linear
- Toricelli (not on midterm) – separable
- input – output modeling – linear, sometimes separable
- population models – separable
- improved velocity models – linear drag → linear DE; quadratic drag → separable.

3a) *For functions* $y(x)$, *why is*

$$L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$$

called <u>*linear*</u>?

(1) $L(y_1+y_2) = L(y_1) + L(y_2)$

& (2) $L(cy) = c\,L(y)$ (c constant)

(so $L(c_1y_1 + c_2y_2 + \dots + c_ky_k) = c_1L(y_1) + \dots + c_kL(y_k)$)

3b) *For linear operators L*, why is the general solution to

$$L(y) = f$$

given by $y = y_p + y_H$, where y_p is any single particular solution, and y_H is the general solution to the homogeneous problem?

Because if

$L(y_p) = f$ and $L(y_H) = 0$

then $L(y_p + y_H) = L(y_p) + L(y_H)$

$= f + 0 = f$

And if also $L(y_q) = f$

then $y_q = y_p + (y_q - y_p)$

and $L(y_q - y_p) = L(y_q) - L(y_p)$

$= f - f = 0$

so $y_q - y_p$ is a y_H, homog. sol.

3c) *For the differential operator L above, what is the dimension of the solution space to the homogeneous DE*

$$L(y) = 0?$$

What does this have to do with the existence-uniqueness theorem?

dimension $= n$, because each IVP has a unique solution, so any set of n solutions whose Wronskian matrix at x_0 is invertible will be a basis

$$\begin{cases} L(y) = 0 \\ y(x_0) = b_1 \\ y'(x_0) = b_2 \\ \vdots \\ y^{(n-1)}(x_0) = b_n \end{cases}$$

3d) *Can you check whether collections of functions are linearly independent?*

I can. One way is to use the Wronskian matrix

$$W(y_1, y_2, \dots, y_n)(x) = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}.$$

If it's invertible at any x_0 (e.g. if its determinant, "the Wronskian" is non-zero there), then the functions are linearly independent

3e) What's the Wronskian matrix? How does it arise in studying initial value problems?

e.g. if $n=2$ and we're studying $\begin{cases} y'' + p(x)y' + q(x)y = f \\ y(x_0) = b_1 \\ y'(x_0) = b_2 \end{cases}$

and if $y = y_P + y_H$

$y = y_P + c_1 y_1 + c_2 y_2$

then $y' = y_P' + c_1 y_1' + c_2 y_2'$

so at x_0 we need to solve the matrix equation $\begin{bmatrix} y_P(x_0) \\ y_P'(x_0) \end{bmatrix} + \underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{\text{Wronskian matrix at } x_0} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

3f) What's the algorithm for finding the solution space to

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

(when all the a_j are constants)? What is Euler's formula, and what does it have to do with this discussion? How are repeated roots to the characteristic polynomial handled? Why are the solutions that the algorithm creates linearly independent?

try for a basis made of exponential fcns and their relatives.

compute $L(e^{rx}) = e^{rx}\underbrace{(r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0)}_{p(r) \text{ "characteristic polynomial"}}$

for every root r of $p(r)$, we get a solution $y = e^{rx}$

If $r = a \pm bi$ we get sol'tns $e^{ax}\cos bx$, $e^{ax}\sin bx$

If roots are repeated k times we also get $xe^{rx}, \ldots x^{k-1}e^{rx}$ (or $xe^{ax}\cos bx$, $xe^{ax}\sin bx, \ldots$ $x^{k-1}e^{ax}\cos bx$, $x^{k-1}e^{ax}\sin bx$)

3g) For the application to unforced (but possibly damped) mass-spring configurations

$$m\,x''(t) + c\,x'(t) + k\,x(t) = 0$$

what sorts of phenomena arise? Can you convert to amplitude-phase form for simple harmonic motion? Can you describe the important quantities for simple harmonic motion? How are damping phenomena classified? Can you solve IVPs?

undamped — $c = 0$ — $x(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \alpha)$ simple harmonic motion

$p(r) = r^2 + \frac{k}{m} = 0$ $\qquad \omega_0 = \sqrt{\frac{k}{m}}$

$r = \pm i\sqrt{\frac{k}{m}}$.

underdamped: roots of $p(r)$ are complex } $x(t)$ oscillates ∞'ly often but decays exponentially

critically damped: double real root, which is negative

overdamped: two negative real roots

} $x(t)$ decays exponentially without oscillating

- be able to find amplitude, phase angle, time delay for simple harmonic motion

also: angular frequency
frequency
period