Wolfram alpha: It will solve any of the DE IVP's on the previous pages, for example the underdamped one:

x"(t)+2*x'(t)+9*x(t)=0, x(0)=1, x'(0)=3/2

Browse Examples   Surprise Me

Differential equation solution:

$$x(t) = \frac{1}{8} e^{-t} \left(5\sqrt{2}\sin\left(2\sqrt{2}\,t\right) + 8\cos\left(2\sqrt{2}\,t\right)\right)$$

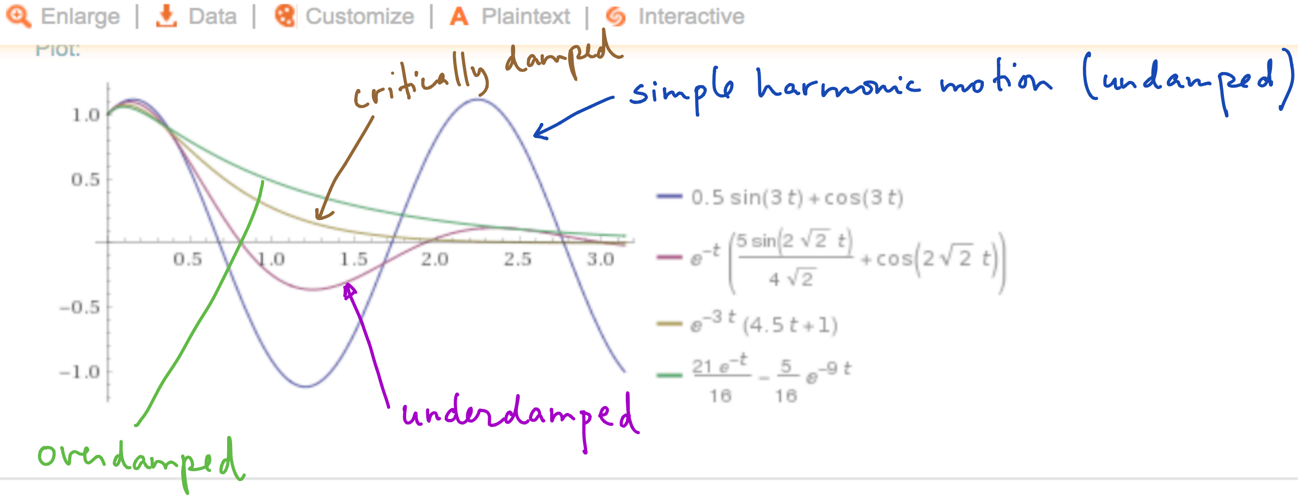
Display of graphs of all 4 solution functions:

Input interpretation:

plot	$\cos(3t) + 0.5\sin(3t)$	$t = 0$ to π
	$\exp(-t)\left(\frac{5}{8}\sqrt{2}\sin\left(2\sqrt{2}\,t\right) + \cos\left(2\sqrt{2}\,t\right)\right)$	
	$\exp(-3t)(1 + 4.5t)$	
	$\frac{21}{16}\exp(-t) - \frac{5}{16}\exp(-9t)$	

Enlarge | Data | Customize | Plaintext | Interactive

Plot:



Tues Feb 12:
3.4 continued....systematic summary of the physical phenomena associated with unforced damped mass-spring configurations.

Announcements:
- Exam review is 1:00 – 2:20 Thursday
  it's in LCB 323
- Exam is 12:50 – 1:50 on Friday.
- last 10 minutes on handout, related to last HW problem.

(HW)

Warm-up Exercise: Use Euler's formula and the angle addition formulas for $\cos(\alpha+\beta)$ & $\sin(\alpha+\beta)$ to verify the rule of exponents

$$e^{i(\alpha+\beta)} = e^{i\alpha+i\beta} = e^{i\alpha}e^{i\beta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = -1 \quad \heartsuit$$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$+ i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

$m x'' + c x' + k x = 0$ $\quad x(t)$

systematically with letters

Case 1) no damping ($c = 0$).

$$m x'' + k x = 0$$

$$x'' + \frac{k}{m} x = 0 .$$

$x = e^{rt}$

$L(x) = e^{rt} p(r)$

$$p(r) = r^2 + \frac{k}{m} ,$$

has purely imaginary roots

$e^{a \pm ibt}$ cplx soln,

$\Rightarrow e^{at}\cos bt,\ e^{at}\sin bt$ real fcn sol's

$$r^2 = -\frac{k}{m} \quad \text{i.e.} \quad r = \pm i \sqrt{\frac{k}{m}} .$$

So the general solution is

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right) .$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the natural angular frequency. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2} x = 0$$

is

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) .$$

"simple harmonic motion"

It's worth learning to recognize the undamped DE, and the trigonometric solutions, as it's easy to understand why they are solutions and you can then skip the characteristic polynomial step.

Exercise 1a Write down the general homogeneous solution $x(t)$ to the differential equation

$$x''(t) + 4\, x(t) = 0.$$

$\sqrt{4} = 2$

$x(t) = A\cos 2t + B\sin 2t$

1b) What is the general solution to $\theta(t)$ to

$$\theta''(t) + 10\, \theta(t) = 0.$$

$\theta(t) = A\cos\sqrt{10}\,t + B\sin\sqrt{10}\,t$

The motion exhibited by the solutions

$$\boxed{x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)}$$

to the undamped oscillator DE

$$x''(t) + \omega_0^{\,2}\, x(t) = 0$$

is called <u>simple harmonic motion.</u> The reason for this name is that $x(t)$ can be rewritten in "amplitude-phase form" as

$$x(t) = \boxed{C\cos(\omega_0 t - \alpha)} = C\cos(\omega_0(t - \delta))$$

in terms of an <u>amplitude</u> $C > 0$ and a <u>phase angle</u> α (or in terms of a <u>time delay</u> δ).

To see why this is so, equate the two forms and see how the coefficients A, B, C and phase angle α must be related:

warm-up

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\left[\cos\omega_0 t \cos(-\alpha) - \sin\omega_0 t \cdot \sin(-\alpha)\right]$$

$$= C\cos\alpha \cos\omega_0 t + C\sin\alpha \sin\omega_0 t$$

<u>Exercise 2)</u> Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to show that the two expressions above are equal provided

$$A = C\cos\alpha$$
$$B = C\sin\alpha$$

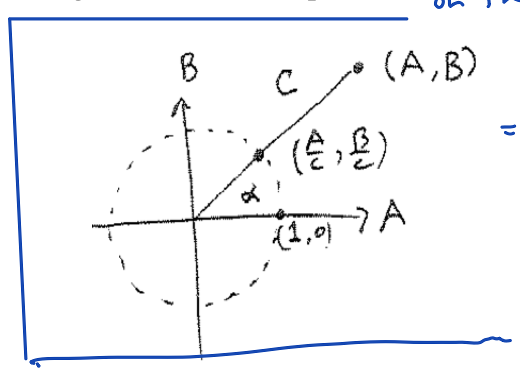
so $A^2 + B^2 = C^2\cos^2\alpha + C^2\sin^2\alpha$
$= C^2(\cos^2\alpha + \sin^2\alpha) = C^2$

So if C, α are given, the formulas above determine A, B. Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha),\ \frac{B}{C} = \sin(\alpha)$$

$\begin{bmatrix} \frac{A}{C} \\ \frac{B}{C} \end{bmatrix}$ is a unit vector since $C = \sqrt{A^2+B^2}$, so it's on the unit circle. So $= \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$

determine C, α. These correspondences are best remembered using a diagram in the $A - B$ plane:



It is important to understand the behavior of the functions

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t-\delta))$$

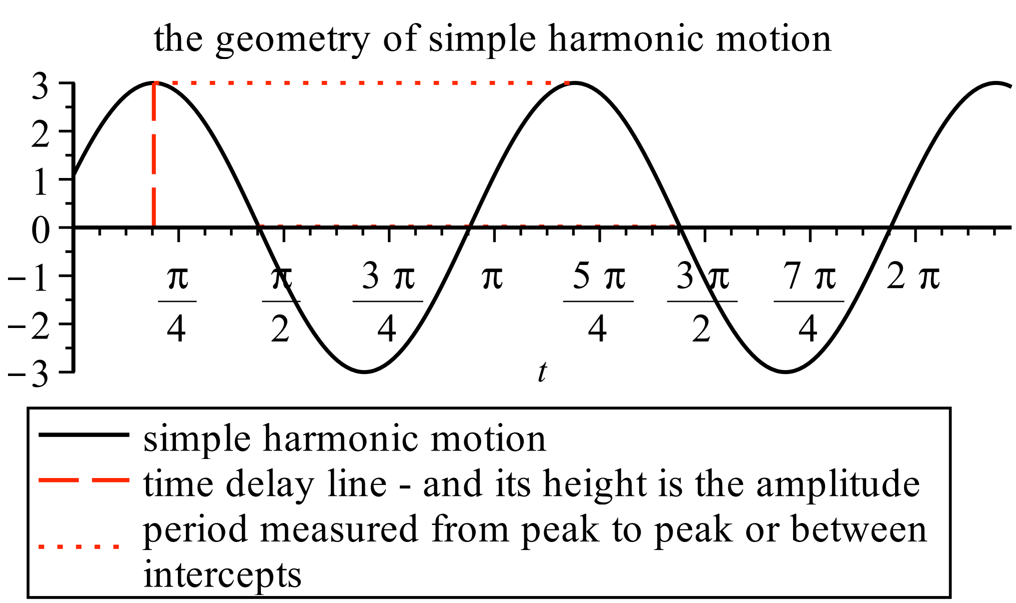
and the standard terminology:

The amplitude C is the maximum absolute value of $x(t)$. The *phase angle* α is the radians of $\omega_0 t$ on the unit circle, so that $\cos(\omega_0 t - \alpha)$ evaluates to 1. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right along the t-axis in order to obtain the graph of $x(t)$. Note that

ω_0 = angular velocity units: radians/time

f = frequency $= \dfrac{\omega_0}{2\pi}$ units: cycles/time

T = period $= \dfrac{2\pi}{\omega_0}$ units: time/cycle.



Exercise 3) Yesterday we solved the differential equation IVP

$$x'' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$\omega_0^2 = 9, \ \omega_0 = 3$

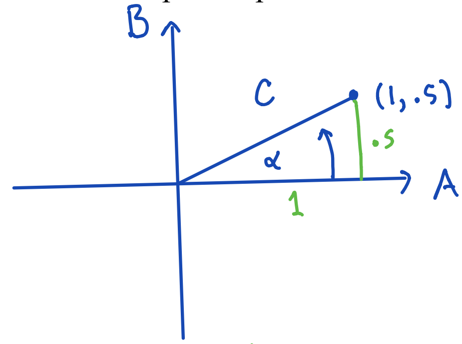
Its solution is

$$x(t) = \cos(3t) + \frac{1}{2}\sin(3t). \quad = C\cos(3t - \alpha)$$

$A = 1 \qquad B = .5$

Convert the formula for $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the Wolfram alpha output on the next page

$C = \sqrt{1 + .25} = \sqrt{1.25} \cong 1.12$



$\tan\alpha = .5$

$\cos\alpha = \frac{1}{C} = \frac{1}{\sqrt{1.25}} = \frac{2}{\sqrt{5}}$

$\sin\alpha = \frac{.5}{C} = \frac{.5}{\sqrt{1.25}}$

$\alpha \cong .46$

in this example α is in 1^{st} quadrant

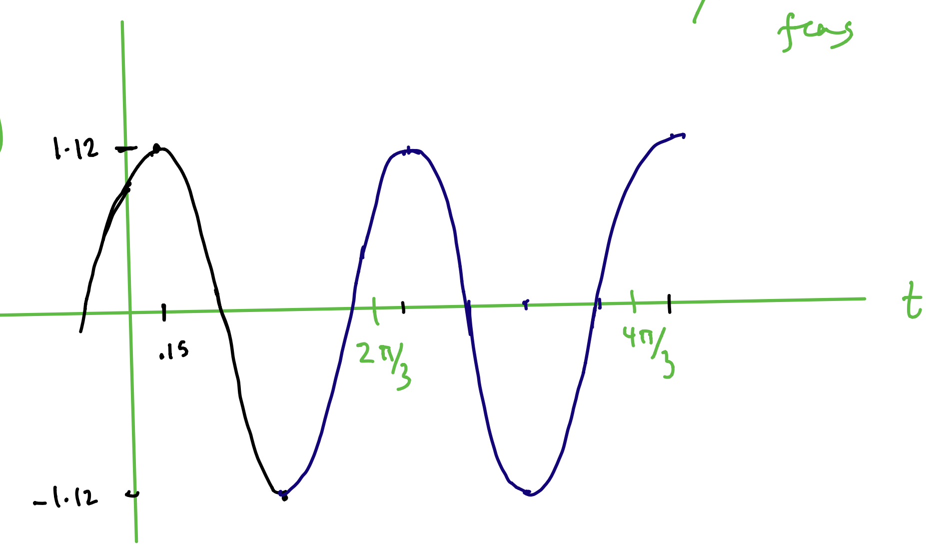
So $\alpha = \tan^{-1}(.5) = \arccos\left(\frac{1}{\sqrt{1.25}}\right) = \arcsin\left(\frac{.5}{\sqrt{1.25}}\right)$

$x(t) \cong 1.12\cos(3t - .46)$
$\cong 1.12\cos(3(t - .15))$

if $\begin{bmatrix} A/C \\ B/C \end{bmatrix}$ is in another quadrant, use more care with your inverse trig fcns

period: $\frac{2\pi}{3} \approx 2$

$t = .15$ time delay



$\arctan(.5) \approx .464$

x"(t)+9*x(t)=0, x(0)=1, x'(0)=3/2

Web Apps Examples Random

Input:

$$\left\{x''(t) + 9\,x(t) = 0,\ x(0) = 1,\ x'(0) = \frac{3}{2}\right\}$$

Open code

Autonomous equation:

$$x''(t) = -9\,x(t)$$

Autonomous equation »

ODE classification:

second-order linear ordinary differential equation

Alternate forms:

$$\left\{x''(t) = -9\,x(t),\ x(0) = 1,\ x'(0) = \frac{3}{2}\right\}$$

$$\{x''(t) + 9\,x(t) = 0,\ x(0) = 1,\ 2\,x'(0) = 3\}$$

Differential equation solution:

Step-by-step solution

$$x(t) = \frac{1}{2}\sin(3\,t) + \cos(3\,t)$$

WolframAlpha computational knowledge engine.

plot cos(3*t)+.5*sin(3*t),t=0..Pi

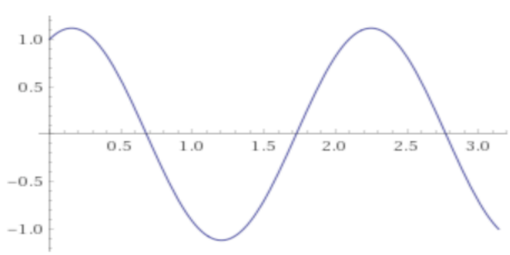
Web Apps Examples Random

Input interpretation:

plot	$\cos(3\,t) + 0.5\sin(3\,t)$	$t = 0$ to π

Open code

Plot:



Case 2: Unforced mass-spring system with damping: (We did concrete examples of each of the three subcases below yesterday.)

- 3 possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

$$m\,x'' + c\,x' + k\,x = 0$$

$$x'' + \frac{c}{m}\,x' + \left(\frac{k}{m}\right)x = 0$$

rewrite as

$$x'' + 2\,p\,x' + \omega_0^2\,x = 0.$$

$\left(p = \frac{c}{2\,m},\ \omega_0^2 = \frac{k}{m}\right)$. The characteristic polynomial is

$$r^2 + 2\,p\,r + \omega_0^2 = 0$$

which has roots

$$r = -\frac{2\,p \pm \sqrt{4\,p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

Case 2a) $(p^2 < \omega_0^2$, or $c^2 < 4\,m\,k)$ underdamped. Complex roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i\,\omega_1$$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$, the undamped angular frequency.

$$x(t) = e^{-p\,t}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) = e^{-p\,t}C\cos(\omega_1 t - \alpha_1).$$

- solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p\,t}C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1.

$$r^2 + 2\,p\,r + \omega_0^2 = 0$$

has roots

$$r = -\frac{2\,p \pm \sqrt{4\,p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}\,.$$

Case 2b) ($p^2 = \omega_0^2$, or $c^2 = 4\,m\,k$) critically damped. Double real root $r_1 = r_2 = -p = -\dfrac{c}{2\,m}$.

$$x(t) = e^{-p\,t}\left(c_1 + c_2\,t\right)\,.$$

- solution converges to zero exponentially fast, passing through $x = 0$ at most once. The critically damped case is the transition between underdamped and overdamped:

Case 2c) ($p^2 > \omega_0^2$, or $c^2 > 4\,m\,k$). overdamped. In this case we have two negative real roots

$$r_1 = -p - \sqrt{p^2 - \omega_0^2} < 0$$

$$r_1 < r_2 = -p + \sqrt{p^2 - \omega_0^2} < 0$$

and

$$x(t) = c_1 e^{r_1 t} + c_2\,e^{r_2 t} = e^{r_2 t}\left(c_1 e^{(r_1 - r_2)\,t} + c_2\right)\,.$$

- solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once, just like in the critically damped case. We did a specific example of all possible cases yesterday, and it may help to review the final picture at the end of Monday's notes.

Math 2280-002
February 12, 2019

Magic worksheet
(to help with last homework problem this week)

From Math 2270:

1) If $S, T : V \rightarrow W$ are linear transformations, then you can add them and scalar multiply them to get new linear transformations:

$$(S + T)(\underline{\boldsymbol{v}}) := S(\underline{\boldsymbol{v}}) + T(\underline{\boldsymbol{v}})$$
$$(c\,S)(\underline{\boldsymbol{v}}) := c\,S(\underline{\boldsymbol{v}}).$$

2) If $T_1 : V \rightarrow W$ and $T_2 : W \rightarrow Z$ are linear transformations, then so is the composition $T_2 \circ T_1 : V \rightarrow Z$.

application to 2280:

3) Let

$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + \ldots. + a_1 y' + a_0 y$$

be the constant coefficient linear operator on the left side of our Chapter 3 linear differential equations. Let

$$\mathrm{D}(y) := y'$$

be the derivative operator. The we may express L in terms of D as

$$L = \mathrm{D}^n + a_{n-1}\mathrm{D}^{n-1} + \ldots + a_1 \mathrm{D} + a_0 I$$

$L(y) = D^n y + a_{n-1} D^{n-1} y + \ldots$

$=$ same as above

where I is the identity operator, and $\mathrm{D}^k = \mathrm{D} \circ \mathrm{D} \ldots \circ \mathrm{D}$, k times.

4) Let

$$L(y) = y'' - y' - 12\,y$$
$$L = D^2 - D - 12\,I$$

a) Compute the characteristic polynomial for the differential equation

$$y'' - y' - 12\,y = 0.$$

$$p(r) = r^2 - r - 12 = (r+3)(r-4)$$

$\left(\text{so } e^{-3x}, e^{4x} \text{ solve } L(y) = 0\right)$

b) Related to your computation in part a), show that L can be written as a composition,

$$L = (D - 4\,I) \circ (D + 3\,I) = (D + 3\,I) \circ (D - 4\,I).$$

$$= D \circ (D + 3I) - 4I \circ (D + 3I)$$
$$= D^2 + 3D - 4D - 12I$$
$$= D^2 - D - 12I \checkmark$$

c) Show that

$$(D - 4\,I)e^{4\,x} = 0, \qquad (D + 3\,I)e^{-3\,x} = 0.$$

This is why $\{e^{4\,x}, e^{-3\,x}\}$ are a basis for the solution space to

$$y'' - y' - 12\,y = 0$$

$$(D - 4I)\,e^{4x} = 4e^{4x} - 4e^{4x} = 0$$

so $(D+3I) \circ \underbrace{(D - 4I)\,e^{4x}}_{0} = 0$

similarly, $(D - 4I) \circ (D + 3I)\,e^{-3x} = (D - 4I)\,0 = 0.$

I'll fill this in

5) Now let

$$L(y) = y'' - 10\,y' + 25\,y$$
$$L = \mathrm{D}^2 - 10\,\mathrm{D} + 25\,I$$

a) Compute the characteristic polynomial for

$$y'' - 10\,y' + 25\,y = 0$$

and deduce how to factor L into a composition of first order differential operators.

$$p(r) = r^2 - 10r + 25 = (r-5)^2$$

b) Show that

$$(\mathrm{D} - 5\,I)\mathrm{e}^{5\,x} = 0$$
$$(\mathrm{D} - 5\,I) f(x)\mathrm{e}^{5\,x} = f'(x)\mathrm{e}^{5\,x}.$$

$$(D-5I)\,e^{5x} = (e^{5x})' - 5e^{5x} = 5e^{5x} - 5e^{5x} = 0$$
$$(D-5I)\,f(x)e^{5x} = D_x\left(f(x)e^{5x}\right) - 5f(x)e^{5x}$$
$$= f'(x)e^{5x} + \cancel{f(x)5e^{5x}} - \cancel{5f(x)e^{5x}}$$
$$= f'(x)e^{5x}$$

c) Deduce that

$$(\mathrm{D} - 5\,I) \circ (\mathrm{D} - 5\,I)\, x\,\mathrm{e}^{5\,x} = 0$$

This is why $\{\mathrm{e}^{5\,x}, x\,\mathrm{e}^{5\,x}\}$ are a basis for the solution space to $L(y) = 0$.

$$(D-5I)\,e^{5x} = 0$$
$$(D-5I)\circ \underbrace{(D-5I)\,x\,e^{5x}}_{1\cdot e^{5x}} = (D-5I)\,e^{5x} = 0$$

d) If the characteristic polynomial of some high order linear homogeneous DE has a factor of $(r-5)^3$, so that the operator has a composition factor of

$$(\mathrm{D} - 5\,I) \circ (\mathrm{D} - 5\,I) \circ (\mathrm{D} - 5\,I) = (\mathrm{D} - 5\,I)^3$$

explain why $\mathrm{e}^{5\,x}, x\,\mathrm{e}^{5\,x}, x^2\mathrm{e}^{5\,x}$ are all solutions to that homogeneous linear DE.

$$(D-5I)^3 f(x)e^{5x} = f'''(x)e^{5x}$$

so $(D-5I)^3 (c_1 + c_2x + c_3x^2)e^{5x}$

$= 0\,e^{5x}$ since $(c_1 + c_2x + c_3x^2)''' = 0.$