Math 2280-002
Friday Feb 1
3.1 Second order linear differential equations and vector space theory connections to Math 2270

Announcements:
- I'll be in Math Student Center computer lab for at least an hour after class today, if want to do your Matlab hw there. (Also Monday 10:45-11:35, Tuesday 2:00-3:00 office hours)
- Computer lab directions handout.

12:57

Warm-up Exercise: Consider the differential equation for $y(x)$:

$$y'' - 2y' - 3y = 0$$

revisited during class: for $L(y) := y'' - 2y' - 3y$ the DE can be written as $L(y) = 0$

a) Show that

$$y_1(x) = e^{3x}, \quad y_2(x) = e^{-x}$$

each solve this DE

linear combination b) How about $y(x) = c_1 e^{3x} + c_2 e^{-x}$, $c_1, c_2 \in \mathbb{R}$?

a) $(e^{3x})'' - 2(e^{3x})' - 3e^{3x} = 9e^{3x} - 2(3e^{3x}) - 3e^{3x}$ $\qquad (e^{3x})' = 3e^{3x}$, $(e^{3x})'' = 9e^{3x}$

$L(e^{3x}) = 0$ $\qquad = e^{3x}(9 - 6 - 3) \equiv 0$

$(e^{-x})'' - 2(e^{-x}) - 3e^{-x} = e^{-x} - 2(-e^{-x}) - 3e^{-x}$

$L(e^{-x}) = 0$ $\qquad = e^{-x}(1 + 2 - 3) \equiv 0$

b) $(c_1 e^{3x} + c_2 e^{-x})'' - 2(c_1 e^{3x} + c_2 e^{-x})' - 3(c_1 e^{3x} + c_2 e^{-x})$

$(9c_1 e^{3x} + c_2 e^{-x}) \quad -2(3c_1 e^{3x} - c_2 e^{-x}) \quad -3(c_1 e^{3x} + c_2 e^{-x})$

$c_1 e^{3x}(9 - 6 - 3)$

$+ c_2 e^{-x}(1 + 2 - 3)$ $\qquad = 0 + 0 \equiv 0!$

revisited during class:

$$L(c_1 e^{3x} + c_2 e^{-x}) = c_1 L(e^{3x}) + c_2 L(e^{-x})$$
$$= c_1 \cdot 0 + c_2 \cdot 0 = 0$$

3.1 Second order linear differential equations, and vector space theory connections to Math 2270:

2270 !! Chapter 4 of LinAlg text, function vector spaces in Chapter 6, at end.

Definition: A <u>vector space</u> is a collection of objects together with an "addition" operation "+", and a "scalar multiplication" operation, so that the rules below all hold.

(α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)
(β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

As well as:

(a) $f + g = g + f$ (commutative property) $(f+g)(x) := f(x)+g(x) = g(x)+f(x) = (g+f)(x)$ number addition is commutative
(b) $f + (g + h) = (f + g) + h$ (associative property)
(c) $\exists\, 0 \in V$ so that $f + 0 = f$ is always true. ← zero function $0(x) \equiv 0$
(d) $\forall f \in V\ \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses)
(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)
(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)
(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)
(h) $1 \cdot f = f$, $(-1) \cdot f = -f$, $0 \cdot f = 0$ (these last two actually follow from the others). ↑ vector

Examples you've seen in Math 2270:
(1) $\mathbb{R}^m$, with the usual vector addition and scalar multiplication, defined component-wise
(2) subspaces W of $\mathbb{R}^m$, which satisfy (α),(β), and therefore automatically satisfy (a)-(h), because the vectors in W also lie in $\mathbb{R}^m$.

Maybe you've also seen ...

<u>Exercise 1</u>) In Chapter 3 we focus on the <u>vector space</u>

$$V = C(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \ s.t.\ f \text{ is a continuous function} \}$$

and its subspaces. Verify that the vector space axioms for linear combinations are satisfied for this space of functions. Recall that the <u>function $f + g$ is defined</u> by $\boxed{(f + g)(x) := f(x) + g(x)}$ and the scalar multiple $c f(x)$ is defined by $\boxed{(c f)(x) := c f(x).}$ What is the <u>zero vector for functions</u>?

↑ the function which is zero for all x

• <u>Basis</u> for vector space V is a set of vectors $\{f_1, f_2, \dots f_n\}$ so that
(i) $\text{span}\{f_1, f_2, \dots f_n\} = V$
(ii) $\{f_1, f_2, \dots f_n\}$ is linearly independent

• Dimension of vector space V is # of vectors in <u>any</u> basis, e.g. $\dim(\mathbb{R}^3) = 3$

2270 • a linear combination of $f_1, f_2, \ldots f_n$ is any sum of scalar multiples of $f_1 \ldots f_n$ i.e. any $f = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$

Recall that the vector space axioms are exactly the arithmetic rules we use to work with linear combination equations. In particular the following concepts are defined in any vector space V.

- the span of a finite collection of functions $f_1, f_2, \ldots f_n$.
- linear independence/dependence for a collection of functions $f_1, f_2, \ldots f_n$.
- subspaces of V
- bases and dimension for finite dimensional subspaces. (The function space V itself is infinite dimensional, meaning that no finite collection of functions spans it.)

• span$\{f_1, f_2, \cdots f_n\}$ = collection of all linear combinations of $f_1, \ldots f_n$
$= \{c_1 f_1 + c_2 f_2 + \cdots + c_n f_n \mid c_1, c_2, \cdots c_n \in \mathbb{R}\}$

• $\{f_1, f_2, \cdots f_n\}$ is linearly independent the only way $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ is if $c_1 = c_2 = \cdots = c_n = 0$

compact way of saying that no f_i is a linear combination of some of the others

• W is a sub (vector) space of V means the subset W is closed under addition and under scalar multiplication i.e.
α) $f, g \in W \Rightarrow f + g \in W$
β) $f \in W, c \in \mathbb{R} \Rightarrow cf \in W$

Definition: A second order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$\boxed{y'' + p(x)y' + q(x)y = f(x).}$$

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

Definition: The DE above is called homogeneous if the right hand side $f(x)$ is the zero function, $f(x) \equiv 0$. If f is not the zero function, the DE is called nonhomogeneous (or inhomogeneous).

2270
$A\vec{x} = \vec{0}$ "homogeneous"

One reason the DE above is called linear is that the "operator" L defined by

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$
(2) $L(cy) = cL(y)$, $c \in \mathbb{R}$.

• See warmup for concrete example

(Recall that the matrix multiplication function $L(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ satisfies the analogous properties. Any time we have have a transformation L satisfying (1),(2), we say it is a linear transformation.)

Exercise 2a) Check the linearity properties (1),(2) for the differential operator $L(y) := y'' + p(x)y' + q(x)y$.

1st order linear DE for $y(x)$

$$\boxed{y' + p(x)y = f(x)}$$

(1) $L(y_1 + y_2) = (y_1 + y_2)'' + p(x)(y_1 + y_2)' + q(x)(y_1 + y_2)$

$= (y_1'' + y_2'') + p(x)(y_1' + y_2') + q(x)(y_1 + y_2)$

$= (y_1'' + p(x)y_1' + q(x)y_1) + (y_2'' + p(x)y_2' + q(x)y_2)$

$= L(y_1) + L(y_2)$

(2) $L(cy)$
$= (cy)'' + p(x)(cy)' + q(x)(cy)$
$= c[y'' + p(x)y' + q(x)y]$

$= c\, L(y)$

2270: Nul A

solution space to eqtn $A\vec{x} = \vec{0}$

2b) Use these linearity properties to show that

Theorem 0 the solution space to the homogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = 0$$

is closed under addition and scalar multiplication, i.e. it is a subspace. Notice that this is the "same" proof one uses to show that the solution space to a homogeneous matrix equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$ is a subspace.

See warmup

to be continued...

Exercise 3) As an example, find the solution space to the following homogeneous differential equation for $y(x)$

$$y'' + 2\,y' = 0$$

on the x-interval $-\infty < x < \infty$. Notice that the solution space is the span of two functions. Hint: This is really a first order DE for $v = y'$.

Exercise 4) Use the linearity properties to show

Theorem 1 All solutions to the nonhomogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = f(x)$$

are of the form $y = y_P + y_H$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE.

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Exercise 5) Verify Theorems 1 and 2 for the interval $I = (-\infty, \infty)$ and the IVP

$$y'' + 2y' = 3$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing to solve these differential equations. It will help to know

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional.

This Theorem is illustrated in Exercise 2 that we completed earlier. Theorem 3 and the techniques we'll actually be using going forward are illustrated by

Exercise 6) Consider the homogeneous linear DE for $y(x)$

$$y'' - 2\,y' - 3\,y = 0$$

6a) Find two exponential functions $y_1(x) = e^{r\,x}$, $y_2(x) = e^{\rho\,x}$ that solve this DE. Deduce that arbitrary linear combinations of y_1, y_2 also solve the DE.

6b) Show that every IVP

$$\begin{aligned} y'' - 2\,y' - 3\,y &= 0 \\ y(0) &= b_0 \\ y'(0) &= b_1 \end{aligned}$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

6c) Use your work from part b to explain why the solution space is two-dimensional.

6d) Now consider the nonhomogeneous DE

$$y'' - 2\,y' - 3\,y = 9$$

Notice that $y_P(x) = -3$ is a particular solution. Use this information and superposition (linearity) to find the solution to the initial value problem

$$\begin{aligned} y'' - 2\,y' - 3\,y &= 9 \\ y(0) &= 6 \\ y'(0) &= -2. \end{aligned}$$