Tues April 9

9.1 Introduction to Fourier series – combo of linear algebra & analysis

Announcements:

Warm-up Exercise: Find the projection of $\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \in \mathbb{R}^4$ onto the subspace V spanned by the orthogonal vectors $\left\{ \vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.
(Using Math 2270 formula)

$v \perp w$ iff $\vec{v} \cdot \vec{w} = 0$

$$\text{proj}_V \vec{x} = \left(\frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \right) \vec{v}_1 + \left(\frac{\vec{x} \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \right) \vec{v}_2 + \left(\frac{\vec{x} \cdot \vec{v}_3}{\|\vec{v}_3\|^2} \right) \vec{v}_3$$

$$= \left(\frac{-4}{4} \right) \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \left(\frac{3}{9} \right) \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \\ 0 \end{bmatrix} \quad \text{as expected}$$

$$\text{proj}_V \vec{x} = \left(\vec{x} \cdot \frac{\vec{v}_1}{\|\vec{v}_1\|} \right) \frac{\vec{v}_1}{\|\vec{v}_1\|} + \ldots$$

$$= (\vec{x} \cdot \vec{u}_1) \vec{u}_1 + (\vec{x} \cdot \vec{u}_2) \vec{u}_2 + (\vec{x} \cdot \vec{u}_3) \vec{u}_3$$

why

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + \vec{z} \quad \text{weights } c_1, c_2, c_3 \text{ unknown}, \ \vec{z} \perp V$$

$$\Longrightarrow \quad \vec{x} \cdot \vec{v}_1 = (c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + \vec{z}) \cdot \vec{v}_1 = c_1 \|\vec{v}_1\|^2 + 0 + 0 + 0$$

$$\text{so} \quad c_1 = \frac{\vec{x} \cdot \vec{v}_1}{\|\vec{v}_1\|^2}$$

Dot products and inner products

Flow chart of dot product geometry in $\mathbb{R}^n$:

$$\vec{x}\cdot\vec{y} := \sum_{j=1}^{n} x_j \cdot y_j$$

$\downarrow$ algebra

a) $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$ (symmetry)

b) $\vec{x}\cdot(\vec{y}+\vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}$; $\vec{x}\cdot(c\vec{y}) = c\,\vec{x}\cdot\vec{y}$ (linear in each argument)

c) $\vec{x}\cdot\vec{x} \geq 0$; $\vec{x}\cdot\vec{x} = 0$ iff $\vec{x} = \vec{0}$ (positive)

An <u>inner product space</u> is a (real scalar) vector space V together with an "inner product" $\langle \bullet, \bullet \rangle$ which gives a real number for each pair of vectors, so that the following axioms hold for all $f, g, h \in V$, $c \in \mathbb{R}$

a) $\langle f, g\rangle = \langle g, f\rangle$

b) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$

$\langle f, cg\rangle = c\langle f, g\rangle$

c) $\langle f, f\rangle \geq 0$. $\langle f, f\rangle = 0$ iff $f = 0$.

from algebra... geometry

magnitude (norm)

$\|\vec{x}\|^2 = \vec{x}\cdot\vec{x}$

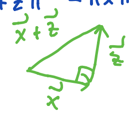
orthogonality

$\vec{x} \perp \vec{z}$ iff $\vec{x}\cdot\vec{z} = 0$

(iff Pythagorean Thm $\|\vec{x}+\vec{z}\|^2 = \|\vec{x}\|^2 + \|\vec{z}\|^2$)

From these algebra axioms the entire concept chart on the left also holds, for finite dimensional subspaces W.

distance from $\vec{x}$ to $\vec{y}$ is $\|\vec{x}-\vec{y}\|$



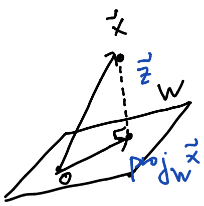
$$\|\vec{x}+\vec{z}\|^2 = (\vec{x}+\vec{z})\cdot(\vec{x}+\vec{z}) = \|\vec{x}\|^2 + \|\vec{z}\|^2 + 2\,\vec{x}\cdot\vec{z}.$$

orthogonal and orthonormal basis for subspaces W

Gram-Schmidt procedure for constructing orthogonal or ortho-normal bases for subspaces

$$\operatorname{proj}_W \vec{x} = \frac{\vec{x}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{x}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}v_2 + \dots + \frac{\vec{x}\cdot\vec{v}_k}{\vec{v}_k\cdot\vec{v}_k}\vec{v}_k$$

is the <u>nearest point to $\vec{x}$</u> in W if $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ is orthogonal basis for W



formula follows by solving for weights $c_1, c_2, \cdots c_k$ so that

$$\vec{z} := \vec{x} - (c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k)$$

is $\perp$ to each $\vec{v}_j$

want

$$0 = \vec{z}\cdot\vec{v}_j = \vec{x}\cdot\vec{v}_j - c_j\,\vec{v}_j\cdot\vec{v}_j \Rightarrow c_j = \frac{\vec{x}\cdot\vec{v}_j}{\vec{v}_j\cdot\vec{v}_j}$$

Examples of function space inner products:

$$V = \{ f : [a, b] \to \mathbb{R} \mid f \text{ is continuous} \} := C([a, b]).$$

$$\langle f, g \rangle := \int_a^b f(t)\, g(t)\, \mathrm{d}t \qquad \text{(or some fixed positive multiple of this integral).}$$

<u>Exercise 1</u>) Check the algebra requirements a), b), c) for an inner product.

a) $\langle f, g \rangle = \langle g, f \rangle$? yes: integrands are $f(t)g(t) = g(t)f(t)$

b) $\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle$?

$$\int_a^b f(t)\left(g(t)+h(t)\right) dt \stackrel{?}{=} \int_a^b f(t)g(t)\, dt + \int_a^b f(t)h(t)\, dt \quad \checkmark$$

$\langle f, cg \rangle = c\langle f, g \rangle$? $\checkmark$

c) $\langle f, f \rangle \geq 0$ $\checkmark$ & $= 0$ iff $f \equiv 0$ $\checkmark$

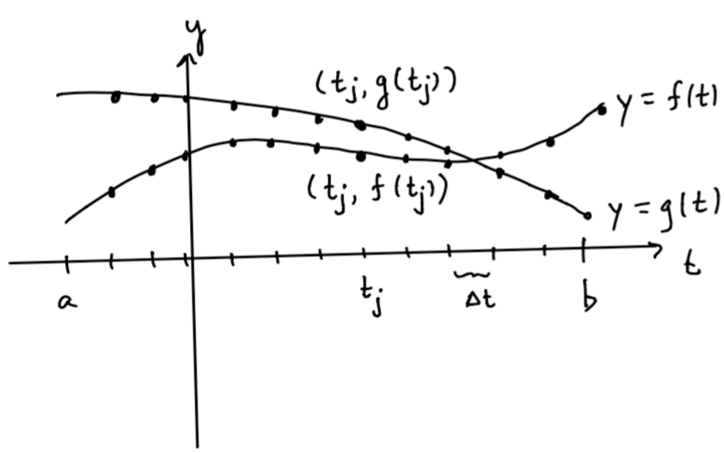
This inner product $\langle f, g \rangle$ is not so different from the $\mathbb{R}^n$ dot product if you think of Riemann sums: Let

$$\Delta t = \frac{b-a}{n}; \qquad t_j = a + j\,\Delta t,\ j = 1, 2, .. n.$$

Then

$$\langle f, g \rangle = \int_a^b f(t) g(t)\, \mathrm{d}t = \lim_{n \to \infty} \sum_{j=1}^{n} f(t_j)\, g(t_j) \Delta t$$

$$= \lim_{n \to \infty} \left(\begin{bmatrix} f(t_1) \\ f(t_2) \\ : \\ f(t_n) \end{bmatrix} \cdot \begin{bmatrix} g(t_1) \\ g(t_2) \\ : \\ g(t_n) \end{bmatrix} \Delta t \right). \quad \frac{b-a}{n}$$



Fourier series

Let $f\colon [-\pi, \pi] \to \mathbb{R}$ be a piecewise continuous function, or equivalently, extend to $f\colon \mathbb{R} \to \mathbb{R}$ as a 2π-periodic, piecewise continuous function.

Then the *Fourier coefficents of* f are computed via the definitions

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t$$

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\,t)\,\mathrm{d}t,\ n \in \mathbb{N}$$

$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t,\ n \in \mathbb{N}$$

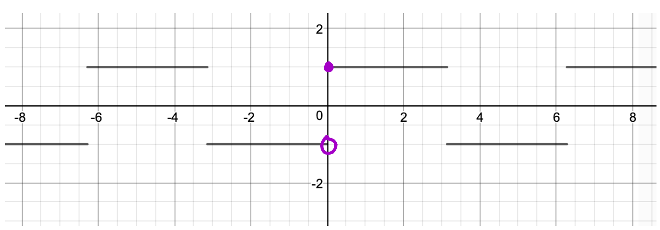
And the *Fourier series for* f is given by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\,t) + \sum_{n=1}^{\infty} b_n\sin(n\,t).$$

The idea is that the partial sums of the Fourier series of f should actually converge to f. The reasons why this should be true combine the linear algebra ideas related to orthogonal basis vectors and projection in the flow chart we just looked over, together with analysis ideas related to convergence. Let's do an example to illustrate the magic, before discussing (parts of) why the convergence actually happens.

Exercise 1 Let's consider the 2π-periodic "square wave" from your canceled homework problem:
The function $sq(t)$ is the 2π-periodic extension of

$$f(t) = \begin{cases} 1 & 0 \le t < \pi \\ -1 & -\pi < t < 0 \end{cases}$$



Compute the Fourier coefficients and Fourier series for $sq(t)$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = 0$$ because the fcn is odd (signed area btwn $-\pi$ & π is zero)

$$\pi a_n = \int_{-\pi}^{\pi} f(t)\cos nt\,dt = \int_{-\pi}^{0} -\cos nt\,dt + \int_{0}^{\pi} \cos nt\,dt = -\frac{\sin nt}{n}\Big]_{0}^{\pi} + \frac{\sin nt}{n}\Big]_{0}^{\pi}$$

$\sin(n\pi) = 0$

$= 0$

$$\pi b_n = \cancel{\frac{1}{\pi}} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \int_{-\pi}^{0} -\sin nt \, dt + \int_{0}^{\pi} \sin nt \, dt$$

$$= \frac{\cos nt}{n}\Big]_{-\pi}^{0} + \frac{-\cos nt}{n}\Big]_{0}^{\pi}$$

$\cos 0 = 1$

$$= \frac{1}{n} - \frac{\cos(-n\pi)}{n} + \frac{-\cos n\pi}{n} + \frac{1}{n}$$

$\cos(-x) = \cos x$

$$= \frac{1}{n}\left(2 - 2\cos n\pi\right)$$

$$\cos n\pi = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

$\cos n\pi = (-1)^n$

so

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$$

$$Sq(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(nt)$$

$$= \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \dots \right)$$

Pictures of the graphs of the partial sums of the first "N" non-zero terms of the Fourier series, stolen from the text. We could've made these graphs with Desmos. Notice how the partial sums are doing their best to approximate the original 2π - periodic square wave we started with.

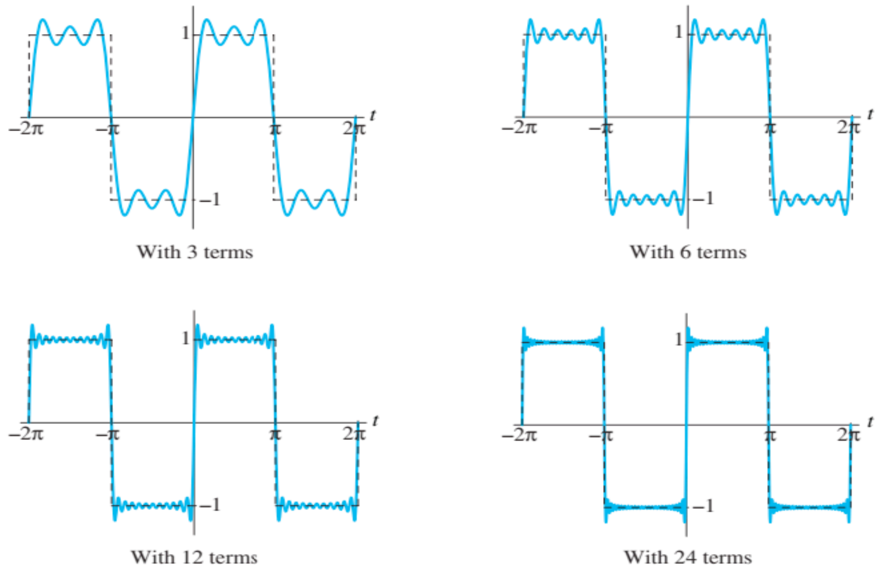


FIGURE 9.1.3. Graphs of partial sums of the Fourier series of the square-wave function (Example 1) with $N = 3, 6, 12$, and 24 terms.

So what's going on?

<u>Theorem</u> Let $V=\{f:\mathbb{R}\to\mathbb{R}\ s.t.\ f$ is piecewise continuous and 2π-periodic$\}$. Define

$$\langle f,g\rangle := \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t.$$

1) Then $V, \langle\ ,\ \rangle$ is an inner product space.
2) Let $V_N := span\{1, \cos(t), \cos(2t), ..., \cos(Nt), \sin(t), \sin(2t), ... \sin(Nt)\}$. Then the $2N+1$ functions listed in this collection are an orthogonal basis for the $(2N+1)$ dimensional subspace V_N. In particular, for any $f\in V$ the nearest function in V_N to f is given by the projection formula

$$proj_{V_N} f = \frac{\langle f,1\rangle}{\langle 1,1\rangle}1 + \sum_{n=1}^{N}\frac{\langle f,\cos(nt)\rangle}{\langle\cos(nt),\cos(nt)\rangle}\cos(nt) + \sum_{n=1}^{N}\frac{\langle f,\sin(nt)\rangle}{\langle\sin(nt),\sin(nt)\rangle}\sin(nt)$$

and this works out to be precisely the truncated Fourier series

$$proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n\cos(nt) + \sum_{n=1}^{N} b_n\sin(nt)$$

where a_0, a_n, b_n are the Fourier coefficients defined earlier:

$$\frac{a_0}{2} := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \frac{\langle f,1\rangle}{\langle 1,1\rangle}$$

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t = \frac{\langle f,\cos(nt)\rangle}{\langle\cos(nt),\cos(nt)\rangle}$$

$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t = \frac{\langle f,\sin(nt)\rangle}{\langle\sin(nt),\sin(nt)\rangle}$$

Exercise 2) Partially check that $\{1,\ \cos(t), \cos(2\,t), ..., \cos(N\,t), \sin(t), \sin(2\,t), ... \sin(N\,t)\}$ is orthogonal for our inner product, and also check why the Fourier coefficients match up to the inner product expressions.

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)g(t)\,\mathrm{d}t$$

Hint:

$$\cos((m+k)t) = \cos(m\,t)\cos(k\,t) - \sin(m\,t)\sin(k\,t)$$
$$\sin((m+k)t) = \sin(m\,t)\cos(k\,t) + \cos(m\,t)\sin(k\,t)$$

so

$$\cos(m\,t)\cos(k\,t) = \frac{1}{2}(\cos((m+k)\,t) + \cos((m-k)t)) \quad \text{(even if } m = k\text{)}$$
$$\sin(m\,t)\sin(k\,t) = \frac{1}{2}(\cos((m-k)\,t) - \cos((m+k)t)) \quad \text{(even if } m = k\text{)}$$
$$\cos(m\,t)\sin(k\,t) = \frac{1}{2}(\sin((m+k)\,t) + \sin((-m+k)t)) \quad \text{(even if } m = k\text{)}$$