Math 2280-002
Week 13, April 8-12, 5.7; 9.1-9.3
Matrix exponential summary and examples; Fourier series for periodic functions

Mon April 8
Matrix exponential summary, and discussion of computations for diagonalizable and non-diagonalizable matrices

Announcements: today: how to compute e^{tA} ("shortcuts").
Tuesday: start Fourier series unit, Chapter 9.

3 examples

Warm-up Exercise: Let $N = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$. Compute $e^{tN} = I + tN + \frac{1}{2!}t^2N^2 + \dots$
Something nice happens...

powers: $N^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow N^3 = [0]$
$N^k = [0] \quad k \geq 2$

$$e^{tN} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} + \cancel{\frac{t^2}{2!}N^2} + \cancel{\dots}$$

$$e^{tN} = \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}$$

DONE!

Def A square matrix "N" is called nilpotent if some power of N, say $N^k = [0]$ the zero matrix.

In that case e^{tN} is easy to compute, because the power series terms are zero eventually

On Friday we computed e^{tA} using diagonalization, for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which had complex eigendata. (We left the last couple of multiplications because time ran out, but I filled those in for the posted notes.) Today we'll work an example which is more akin to your homework, with a diagonalizable matrix having real eigendata. Then we'll see an example of what can happen for non-diagonalizable matrices...you have a simple example in your homework assignment, and I'll also give an overview what happens in general, with most details omitted. The discussion is related to "Jordan Canonical Form" for matrices, which is usually included in the course *Math 5310-5320, Introduction to Modern Algebra.* (See also the Wikipedia page on matrix exponentials, although it's probably too abbreviated to be understandable with respect to Jordan Canonical form, without further digging.) Our text goes into some detail on computing e^{tA} for nondiagonalizable matrices as well, although we won't cover those details in this class.

First, recall/summarize from last week

(1) Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Then

$$e^{A} := I + A + \frac{1}{2!}A^2 + \ldots + \frac{1}{k!}A^k + \ldots$$

$$\left(\Rightarrow e^{tA} := I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^k}{k!}A^k + \ldots. \right)$$

(2) If A and B commute, i.e. $AB = BA$, then

$$e^A e^B = e^B e^A; \qquad e^{A+B} = e^A e^B$$

$$\left(\Rightarrow e^{t(A+B)} = e^{tA} e^{tB} \quad \text{and} \quad \left(e^{tA}\right)^{-1} = e^{-tA} \right)$$

(3)

$$\frac{d}{dt} e^{tA} = A e^{tA} \quad \left(= e^{tA} A \right).$$

In fact, e^{tA} is the unique $n \times n$ matrix $X(t)$ which satisfies

$$X'(t) = AX$$
$$X(0) = I.$$

($X(t) = e^{tA}$ solves this matrix IVP. If any other square matrix function also did, then each of its columns would solve the same IVP's as those of e^{tA}, so would have to be the columns of e^{tA} by uniqueness of solutions to IVP's.)

(4) If

$$tD = \begin{bmatrix} \lambda_1 t & 0 & \dots & 0 \\ 0 & \lambda_2 t & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \lambda_n t \end{bmatrix}$$

then

$$e^{tD} = I + tD + \frac{t^2}{2!}D^2 + \dots + \frac{t^k}{k!}D^k + \dots$$

$$= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2!}\lambda_1^2 t^2 + \dots & 0 & \dots & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2!}\lambda_2^2 t^2 + \dots & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 + \lambda_n t + \frac{1}{2!}\lambda_n^2 t^2 + \dots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}.$$

(5) If A be diagonalizable and $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots, \underline{\boldsymbol{v}}_n\}$ is an eigenbasis for $\mathbb{R}^n$ or $\mathbb{C}^n$ and if P is the invertible matrix with those eigenvectors as columns, and if D is the corresponding the diagonal matrix of eigenvalues so that

$$A P = P D$$
$$A = P D P^{-1} ,$$

Then

$$e^{tA} = I + tP D P^{-1} + \frac{t^2}{2!}\left(PDP^{-1}\right)^2 + \dots + \frac{t^k}{k!}\left(PDP^{-1}\right)^k + \dots$$
$$= P\left(I + tD + \frac{t^2}{2!}t^2 D^2 + \dots \frac{t^k}{k!}t^k D^k + \dots\right)P^{-1}$$
$$= P e^{tD} P^{-1}$$

(6) If the $n \times n$ matrix $\Phi(t)$ is a solution to

$$X'(t) = A\,X$$

(which is true if and only if each column is a solution to the homogeneous system $\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$), and if the columns are a basis for the solution space to $\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$, so that $\Phi(0)$ is invertible, then

$$X(t) = \Phi(t)\,\Phi(0)^{-1}$$

solves the square matrix IVP

$$\begin{aligned} X'(t) &= A\,X \\ X(0) &= I \end{aligned}$$

so

$$\Phi(t)\,\Phi(0)^{-1} = e^{t\,A}.$$

(see (3).)

In this case we call Wronskian $\Phi(t)$ a *Fundamental Matrix Solution (FMS)* for the linear system of differential equations

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}.$$

(7) In the case that A is diagonalizable, we can interpret (5,6) in terms of the Wronskian matrix whose columns are our usual basis for the homogeneous solution space:

$$e^{t\,A} = P\,e^{t\,D}\,P^{-1}$$

$$e^{t\,A} = \begin{bmatrix} | & | & | & | \\ \underline{\mathbf{v}}_1 & \underline{\mathbf{v}}_2 & \cdots & \underline{\mathbf{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \underline{\mathbf{v}}_1 & \underline{\mathbf{v}}_2 & \cdots & \underline{\mathbf{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} | & | & | & | \\ e^{\lambda_1 t}\,\underline{\mathbf{v}}_1 & e^{\lambda_2 t}\,\underline{\mathbf{v}}_2 & \ldots & e^{\lambda_n t}\,\underline{\mathbf{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \underline{\mathbf{v}}_1 & \underline{\mathbf{v}}_2 & \cdots & \underline{\mathbf{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix}^{-1}$$

$$= \Phi(t)\,\Phi(0)^{-1}.$$

cols are basis of solns to $\vec{x}' = A\vec{x}$

$\downarrow$

of $\Phi(t)$

Exercise 1 For the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

the eigendata is

$$E_{\lambda=-2} = span\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\} \qquad E_{\lambda=5} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}.$$

Compute e^{tA}.

use diagonalization first

$$AP = P\underset{\uparrow \text{ evals}}{D} \qquad \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$e^{tA} = P\, e^{tD}\, P^{-1}$$

$$= \underbrace{\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{5t} \end{bmatrix}}\frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c:c} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{array}\right]\frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$e^{tA} = \frac{1}{7}\begin{bmatrix} e^{-2t}+6e^{5t} & -2e^{-2t}+2e^{5t} \\ -3e^{-2t}+3e^{5t} & 6e^{-2t}+e^{5t} \end{bmatrix}$$

$$e^{t\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{5t} \end{bmatrix}$$

$$\| $$

$$I + t\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} (-2)^2 & 0 \\ 0 & (5)^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1+(-2t)+\frac{(-2t)^2}{2!}+\cdots & 0 \\ 0 & 1+5t+\frac{(5t)^2}{2!}+\cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{5t} \end{bmatrix} \quad \text{see (4)}$$

$$P = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

Note $X = e^{tA}$ solves $\begin{cases} X'(t) = AX \\ X(0) = \frac{1}{7}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = I \ \checkmark \end{cases}$

---

text explanation ... focus on system $\vec{x}'(t) = A\vec{x}$: $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$
make F.M.S. of basis soltns

$$= \begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & e^{\lambda_2 t}\vec{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & e^{\lambda_2 t}\vec{v}_2 \end{bmatrix} = \begin{bmatrix} e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} & e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

$\Phi(t)$ solves $\Phi' = A\Phi$, so $\Phi(t)C$ also does $\quad \frac{d}{dt}\left(\Phi(t)C\right) = \Phi'(t)C$
$= A(\Phi C)$
if choose $C = \Phi(0)^{-1}$

then $\Phi(t)C \underset{@ t=0}{=} \Phi(0)\Phi(0)^{-1} = I$

$$\boxed{\text{SO } e^{tA} = \Phi(t)\Phi(0)^{-1}}$$

Same formula as we came up the first way.

Exercise 2 Use the power series definition of matrix exponential to compute e^{tN} for

$$N = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

$$N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{So} \quad e^{tN} = I + tN + \cancel{\tfrac{1}{2!}t^2N^2} + \cancel{\tfrac{1}{3!}t^3N^3} + \cancel{\cdots}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}$$

(warm-up)

Definition A matrix N is called *nilpotent* if some power (and hence all higher powers) of the matrix is the zero matrix. (In fact, if N is an $n \times n$ nilpotent matrix, then it will always be true that $N^n = 0$, although a lower power of N might also be the zero matrix.)

Remark if N is nilpotent then the series for e^{tN} is finite and easy to compute, because after some point all the terms in the series are zero.

Exercise 3a This ties in with Exercise 2, and deals with the only sort of non-diagonalizable case of a 2×2 matrix. Let

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$$

$p(\lambda) = (\lambda + 2)^2$ but $E_{\lambda=-2}$ is defective:

$$E_{\lambda=-2} = span\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}.$$

bad defective eigenspace

$$E_{\lambda=-2}: \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right] = span\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$$

Decompose A as

$$A = -2\,I + N = -2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix},$$

where N is the nilpotent matrix from the previous page. Multiples of the identity matrix commute with all other square matrices of the same size. Use this and your work in Exercise 2 to compute e^{tA}.

$$e^{tA} = e^{-2tI + tN} = e^{t(-2I)}\, e^{tN}$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}$$

$$\left(= e^{-2t}\begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix} \right)$$

Exercise 3b The first order system with matrix A

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2)$$

has solutions $[x(t), x'(t)]^T$ where $x(t)$ solves the critically damped DE

$$x'' + 4x' + 4x = 0. \quad (1)$$

Use this fact to re-compute e^{tA} from part a.

(filled in after class)

- If $x(t)$ solves (1) then $\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -4x-4x' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$

- for $x'' + 4x' + 4x = 0$

  $p(r) = r^2 + 4r + 4 = (r+2)^2$

  so this is critically damped and

  $x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

So sol's to the 1st order system are given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + c_2 t e^{-2t} \\ -2c_1 e^{-2t} + c_2 (e^{-2t} - 2te^{-2t}) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix}}_{\text{this is an FMS } \Phi(t)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$e^{tA} = \Phi(t)\,\Phi(0)^{-1}$$

$$= \begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$\Phi(0) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$\Phi(0)^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$$= e^{-2t} \begin{bmatrix} 1 & t \\ -2 & 1-2t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= e^{-2t} \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}$$

AGREES! with previous page

Although we only treated the 2×2 non-diagonalizable case, there is a theorem which generalizes what we did.....although the statement doesn't really tell you how to construct the decomposition for matrices which are larger than 2×2....this is also related to the Jordan Canonical form decomposition:

<u>Theorem</u> (Jordan–Chevalley decomposition) If A is not diagonalizable, then it is still possible to decompose A as

$$A = B + N$$

where B is diagonalizable, N is nilpotent, and $BN = NB$. Thus

$$e^{tA} = e^{tB + tN} = e^{tB} e^{tN}$$

is straightforward to compute once such a decomposition is known, as in Exercise 3a.

If you're curious, here's more of the story.

Recall from 2270:

<u>Theorem 1</u>: Let $A_{n\times n}$. Let $p(\lambda)=|A-\lambda I| = (-1)^n(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_\ell)^{k_\ell}$

$\lambda_1,\lambda_2,\ldots\lambda_\ell$ distinct

$k_1+k_2+\ldots+k_\ell=n$.

Then $1 \le \dim(E_{\lambda=\lambda_j}) \le k_j$

A is diagonalizable if and only if each $\dim(E_{\lambda=\lambda_j})=k_j$.

i.e. in this case there is an invertible matrix P made out of eigenvector columns so that

$$AP = PD$$

$$D = \begin{bmatrix} \overbrace{\begin{matrix}\lambda_1 & 0\\ 0 & \lambda_1\end{matrix}}^{} \Big\}k_1 & & & 0 \\ & \lambda_2 \; \}k_2 & & \\ 0 & & \ddots & \\ & & & \begin{matrix}\lambda_\ell & 0 \\ 0 & \lambda_\ell\end{matrix}\;\Big\}k_\ell \end{bmatrix}$$

What we probably didn't teach you in 2270:

<u>Theorem 2</u> Let $A_{n\times n}$ as above, but with some defective eigenspaces, i.e.

$$\dim(E_{\lambda=\lambda_j}) < k_j$$

But then the larger "generalized eigenspace" defined as

$$G_{\lambda=\lambda_j} := \mathrm{Nul}\,(A-\lambda_j I)^{k_j}$$

does satisfy

$$\dim(G_{\lambda=\lambda_j}) = k_j.$$

Generalized eigenvector bases can be chosen as the columns of an invertible matrix P so that

$$AP = PJ$$

where J is the "Jordan Canonical Form" of A, made of "Jordan blocks" along the diagonal, and zero elsewhere. Each Jordan block has an eigenvalue along the diagonal, ones along the "superdiagonal" and zeroes elsewhere:

1×1 block: $[\lambda]$, $\quad 2\times2$ block: $\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$ $\quad 3\times3$ block: $\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}$ $\quad 4\times4$: $\begin{bmatrix}\lambda & 1 & 0 & 0\\ 0 & \lambda & 1 & 0\\ 0 & 0 & \lambda & 1\\ 0 & 0 & 0 & \lambda\end{bmatrix}$

etc.

Example Let $A_{4\times 4}$, $|A-\lambda I| = (\lambda-3)(\lambda-2)^3$. The Jordan Canonical form of A (up to ordering of the blocks) is one of:

$$D = \begin{bmatrix} 2&0&0&0\\ 0&2&0&0\\ 0&0&2&0\\ 0&0&0&3 \end{bmatrix}$$

$\dim E_{\lambda=2} = 3$

$\dim E_{\lambda=3} = 1$

A is diagonalizable

$$J = \begin{bmatrix} 2&0&0&0\\ 0&2&1&0\\ 0&0&2&0\\ 0&0&0&3 \end{bmatrix}$$

$\dim E_{\lambda=2} = 2$ (= # of $\lambda=2$ blocks)

$\dim E_{\lambda=3} = 1$

A not diagonalizable

$$J = \begin{bmatrix} 2&1&0&0\\ 0&2&1&0\\ 0&0&2&0\\ 0&0&0&3 \end{bmatrix}$$

$\dim E_{\lambda=2} = 1$

$\dim E_{\lambda=3} = 1$

A not diagonalizable

Every Jordan matrix can be written as

$$J = D + N$$

where D is the diagonal part and N is the superdiagonal part and is nilpotent; and, D commutes with N, $DN = ND$.

$$AP = PJ$$

$$A = PJP^{-1}$$

$$e^{tA} = Pe^{tJ}P^{-1}$$

$$= P\, e^{tD}\, e^{tN}\, P^{-1}$$

e^{tD}: diagonal matrix with entries $e^{t\lambda}$

e^{tN}: finite power series.

Example For $J = \begin{bmatrix} 2&1&0&0\\ 0&2&1&0\\ 0&0&2&0\\ 0&0&0&3 \end{bmatrix}$, $D = \begin{bmatrix} 2&0&0&0\\ 0&2&0&0\\ 0&0&2&0\\ 0&0&0&3 \end{bmatrix}$ $N = \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$

$N^2 = \begin{bmatrix} 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$

$N^3 = [0]$

$$e^{tJ} = e^{tD}\, e^{tN}$$

$$= \begin{bmatrix} e^{2t}&0&0&0\\ 0&e^{2t}&0&0\\ 0&0&e^{2t}&0\\ 0&0&0&e^{3t} \end{bmatrix} \underbrace{\begin{bmatrix} 1&t&t^2/2&0\\ 0&1&t&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}}_{I + tN + \frac{t^2}{2!}N^2}$$