we'll see, the answer relates to syst.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Exercise 1 Use the power series definition to compute e^{tA} for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$e^{tA} = I + t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{t^3}{3!}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} +$$

$$+ \frac{t^4}{4!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{t^5}{5!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \cdots$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$A^3 = AA^2 = -A$$

$$A^4 = AA^3 = A(-A) = -A^2 = I$$

$$= \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots & t - \frac{t^3}{3!} + - \\ & \end{bmatrix}$$

opposites same

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Theorem 1

a)

$$\frac{d}{dt}\, e^{tA} = A\, e^{tA}.$$

b) The j^{th} column of e^{tA} is the solution to the IVP

$$\left.\begin{aligned} \underline{\boldsymbol{x}}'(t) &= A\,\underline{\boldsymbol{x}} \\ \underline{\boldsymbol{x}}(0) &= \underline{\boldsymbol{e}}_j \end{aligned}\right\} \text{ Soltns are unique!}$$

where $\underline{\boldsymbol{e}}_j$ is the standard basis vector which is zero in each entry except for the j^{th} entry, which is 1.

c) The solution to the general homogenous IVP

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

is

$$\underline{\boldsymbol{x}}(t) = e^{tA}\underline{\boldsymbol{x}}_0.$$

Chptr 1

$$\left.\begin{aligned} x'(t) &= ax \\ x(0) &= x_0 \end{aligned}\right\} \; x(t) = x_0 e^{at} \;!!$$

Compare to Chapter 1 for the scalar version!

(c) $e^{tA}\vec{x}_0$ is linear combo of cols, so is sol.

$\leftarrow$ @ $t=0$ reads as $\vec{x}(0) = I\vec{x}_0 = \vec{x}_0$

a) $$\frac{d}{dt}\left(I + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^n}{n!}A^n + \dots\right) = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{t^n}{n!}A^n$$

$$= 0 + A + \underbrace{\frac{2t}{2!}}_{t}A^2 + \dots \frac{t^{n-1}}{(n-1)!}A^n + \dots$$

$0! = 1$

$t^0 = 1$

$A^0 = I$

$$\frac{n\,t^n}{n!}A^n$$

$$= \sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}A^n$$

$$= A\Big[\underbrace{I + tA + \dots + \frac{t^{n-1}}{(n-1)!}A^{n-1}}_{e^{tA}}\Big]$$

$$= A\left(\sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}A^{n-1}\right)$$

$$= A\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k\right)$$

$k = n-1$

b) column by column of e^{At}

$$\frac{d}{dt}e^{At} = \left[\frac{d}{dt}col_1 \;\middle|\; \frac{d}{dt}col_2 \;\middle|\; \cdots \;\middle|\; \frac{d}{dt}col_n\right] = A\left[col_1 \;\middle|\; col_2 \;\middle|\; \cdots\right]$$

So each column satisfies $\vec{x}'(t) = A\vec{x}$.

at $t=0$, $e^{At} = I$

with $\vec{x}(0) = \vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th}$ position

<u>Exercise 2</u> Verify Theorem 1abc for our matrix A in Exercise 1:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad e^{tA} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

(a) $$\frac{d}{dt} e^{At} = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix} \quad \text{same}$$

$$A e^{tA} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix}$$

(b) look column by column

& @ $t=0$ cols are of e^{At} $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

<u>Remark</u>: Theorem 1b indicates another way to compute e^{tA}: Its j^{th} column is the unique solution to the IVP

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{e}}_j$$

We could have used that method to come up with e^{tA} in <u>Exercise 2</u>, especially since the first order system in this case is corresponds to the second order harmonic oscillator differential equation

$$x''(t) + x(t) = 0 \quad (1)$$

There's a different way to compute matrix exponentials for diagonalizable matrices - which we'll discuss tomorrow.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

if $x(t)$ solves (1), then $\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -x \end{bmatrix}$

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$$

Wed April 3
5.6-5.7 Matrix exponentials and fundamental matrix solutions continued.

Announcements:

- pick up HW for next week
- continue matrix exponentials today & Friday
- quiz is a non-complicated mass-spring system problem.

$$e^{i(-t)} = \cos(-t) + i\sin(-t) = \cos t - i\sin t)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Warm-up Exercise: Use Euler's formula to check that

$$\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right) = \frac{1}{2}\left(\cos t + \cancel{i\sin t} + \cos t - \cancel{i\sin t}\right)$$

$$\sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right) = \frac{1}{2i}\left(\cancel{\cos t} + i\sin t - (\cancel{\cos t} - i\sin t)\right)$$

Yesterday we used power series to define matrix exponentials, and saw what e^{tA} has to do with solutions to systems of differential equations and corresponding IVP's

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

If the matrix A is diagonalizable there's a good method to compute e^{tA}, as the next two theorems indicate

<u>Theorem 2</u> If

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

is a diagonal matrix, then

$$tD = t\begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & & \lambda_3 \end{bmatrix} \qquad e^{tD} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

(product of diagonal matrixes is diagonal; entries are products of corresponding diagonal entries

$$e^{Dt} = I + t\begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & & \lambda_3 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} \lambda_1^2 & & O \\ & \lambda_2^2 & \\ O & & \lambda_3^2 \end{bmatrix} + \frac{t^3}{3!}\begin{bmatrix} \lambda_1^3 & & O \\ & \lambda_2^3 & \\ O & & \lambda_3^3 \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 + t\lambda_1 + \frac{1}{2}(t\lambda_1)^2 + \frac{1}{3!}(t_1\lambda_1)^3 + \dots & & O \\ & 1 + t\lambda_2 + \frac{1}{2}(t\lambda_2)^2 + \dots & \\ O & & // \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

Theorem 3 Let A be diagonalizable, i.e.

there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A. Let P be the invertible matrix with those eigenvectors as columns, and let D be the diagonal matrix which has the corresponding eigenvalues in the diagonal entries, i.e.

$$P = [\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n],$$

and

$$A\,P = P\,\mathrm{D}$$
$$A = P\,\mathrm{D}\,P^{-1}$$

Then

$$e^{tA} = P\,e^{t\mathrm{D}}\,P^{-1}$$

$$e^{tA} = \underset{PP^{-1}}{I} + t\underset{PDP^{-1}}{A} + \frac{t^2}{2!}\underset{\substack{PD\cancel{P^{-1}P}DP^{-1} \\ PD^2P^{-1}}}{A^2} + \frac{t^3}{3!}\underset{PD^3P^{-1}}{A^3} + \cdots + \frac{t^n}{n!}\underset{PD^nP^{-1}}{A^n} + \cdots$$

$$= P\left[I + tD + \frac{t^2}{2!}D^2 + \cdots + \frac{t^n}{n!}D^n \qquad\right]P^{-1}$$

$$e^{tA} = P\,\underset{\substack{\uparrow \\ \text{easy}}}{e^{tD}}\,P^{-1}$$

<u>Remark and definition:</u> Group the product expression for e^{tA} as follows:

$$e^{tA} = P\, e^{tD}\, P^{-1}$$

$$e^{tA} = \begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} | & | & | & | \\ e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 & e^{\lambda_2 t}\underline{\boldsymbol{v}}_2 & \ldots & e^{\lambda_n t}\underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \begin{bmatrix} | & | & | & | \\ \underline{\boldsymbol{v}}_1 & \underline{\boldsymbol{v}}_2 & \cdots & \underline{\boldsymbol{v}}_n \\ | & | & | & | \\ | & | & | & | \end{bmatrix}^{-1}$$

$$= \Phi(t)\,\Phi(0)^{-1}.$$

In this case we call $\Phi(t)$ a *Fundamental Matrix Solution (FMS)* for the linear system of differential equations

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}.$$

This is because every solution to the system can be written uniquely as a linear combination of the columns of $\Phi(t)$, i.e. as $\Phi(t)\underline{\boldsymbol{c}}$. We use the same definition in the case that A is not diagonalizable, namely that the columns of $\Phi(t)$ should be a basis for the solution space. And then it will always be true that

$$e^{tA} = \Phi(t)\,\Phi(0)^{-1}.$$