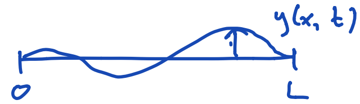
Mon April 22
9.6 wave equation solutions to the natural initial boundary value problems vis Fourier series; traveling wave solutions; slinky math.

Announcements: tomorrow Tuesday will be course review, exam discussion
possible review session time Thursday 6-8 p.m.
we'll decide tomorrow

Warm-up Exercise: review the parts of Friday's notes in today's, so that we can pick up where we left off

On Friday we derived the one space dimension wave equation for transverse vibrations of a vibrating string, and stated the wave equation for parallel (longitudinal) vibrations of a spring. This second model also applies to pressure waves in air, for example sound waves. As another example, earthquakes cause both pressure waves and transverse waves in the earth's crust.



In all models, the function $u(x, t)$ measures the displacement of the point at position x, $0 \le x \le L$, at time t, and one can consider transverse or parallel displacements separately The homogeneous wave equation is the partial differential equation

$$u_{tt} = a^2 u_{xx}$$

where $a > 0$ is a constant which turns out to be *the speed at which waves travel.* Notice that this is a linear homogeneous partial differential equation - the operator $L(u) := u_{tt} - a^2 u_{xx}$ is linear so that the solution space to the homogeneous wave equation is a vector space (linear combinations of solutions are solutions).

We found that for transverse vibrations in the $y-$ direction perpendicular to the string, and applying linearization for the small oscillation model, the displacement functions $y(t)$ satisfy

$$y_{tt} = \frac{T_0}{\rho_0} y_{xx}.$$

This is the wave equation for $y(x, t)$, with

$$y_{tt} = a^2 y_{xx} \ , \quad a = \sqrt{\frac{T_0}{\rho_0}} \ .$$

For parallel (longitudinal) vibrations and if the tension is expressed as a function of the spring density (which is changing depending on time and location), $T = T(\rho)$ then in this case the linearized wave equation turns out to be

$$u_{tt} = b^2 u_{xx}, \qquad b^2 = -T'(\rho_0) > 0$$

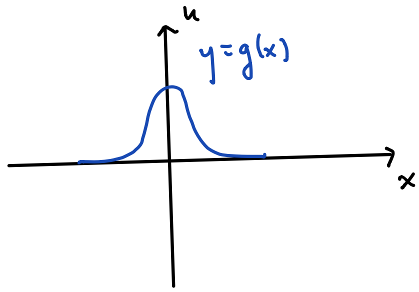
We'll pick up our discussion from Friday....

Special solutions There are two sorts of special solutions for the wave equation: product solutions $u(x,t)=X(x)T(t)$ analogous to the product solutions we used to solve the heat equation initial value problems. And, solutions of the following form, which indicate why we call this the "wave equation".

Exercise 1a For the wave equation

$$u_{tt} = a^2 u_{xx}$$

and for *any* twice continuously differentiable function $g(x)$, use the chain rule to show that $u_1(x,t) = g(x - a\,t)$ and $u_2(x,t) := g(x + a\,t)$ solve the wave equation.



$u(x) = g(x - at)$

$u_t = g'(x-at)(-a)$ Chain rule

$u_{tt} = g''(x-at)a^2$

$u_x = g'(x-at)\cdot 1$

$u_{xx} = g''(x-at)$

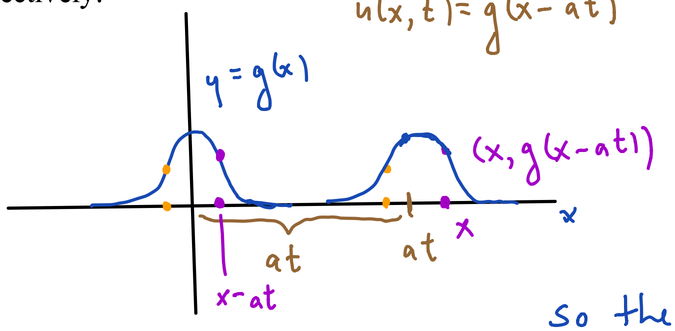
$u_{tt} = a^2 u_{xx}$?

$g''(x-at)a^2 = a^2 g''(x-at)$ ✓

also works for $u(x) = g(x+at)$

1b Interpret these solutions as waves traveling to the right with speed a, and to the left with speed a, respectively.

$u(x,t) = g(x-at)$



original profile, shifted to the right by at

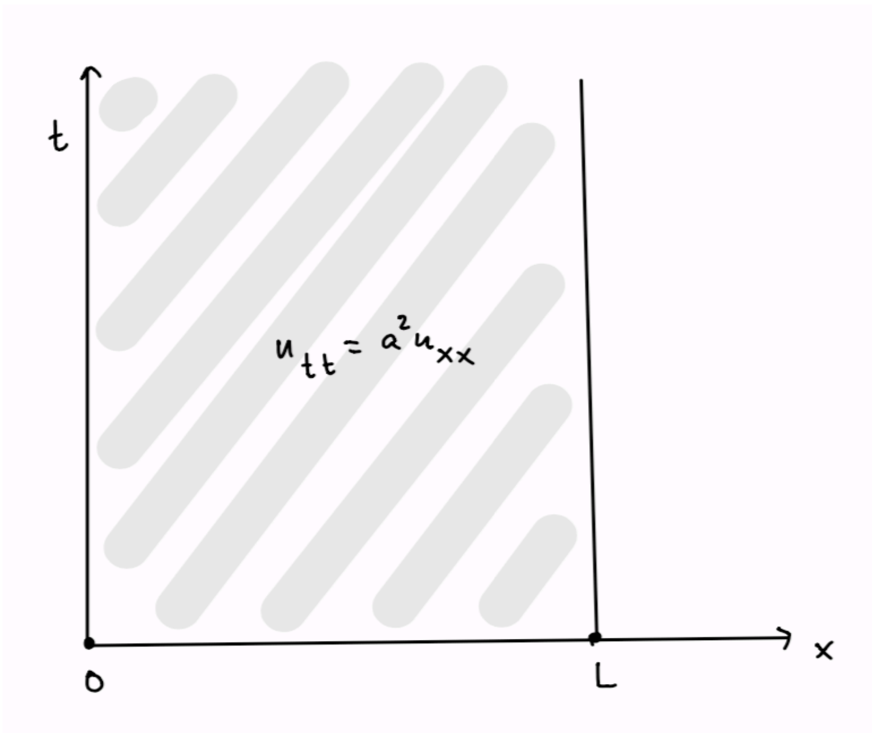
so the profile of $u(x,t)$ is just the profile of $g(x)$ shifted by at i.e moving with speed a

analogous for $u(x,t) = g(x+at)$, except wave moves to the left

1c For transverse vibrations, the speed $a = \sqrt{\dfrac{T_0}{\rho_0}}$. Does this comport with your intuition about how fast waves should travel as we vary the spring tension and density?

yes: higher tension → greater speed
higher density → slower speed

The natural initial value boundary value problems for the wave equation on finite-length intervals:



$$u_{tt} = a^2 u_{xx} \qquad 0 < x < L, t > 0$$

$u(x, 0) = f(x)$ initial displacement

$u_t(x, 0) = g(x)$ initial velocity

with either fixed endpoint or free endpoint boundary conditions:

type 1: fixed endpoints

$$u(0, t) = u(L, t) = 0 \qquad t > 0$$

type 2: free endpoints

$$u_x(0) = u_x(L) = 0 \qquad t > 0$$

On Monday we'll show how to use Fourier series with superposition of product solutions to solve the natural initial value problems, analogously to how we used them for the heat equation. We'll also relate the Fourier series method to the traveling wave method we discussed in Exercise 1, and play with a slinky. If we have time today, we can do the following warm-up exercise:

<u>Exercise 2</u> Let $f(x) = \sin(\omega x)$ (or $f(x) = \cos(\omega x)$). Find a product solution $u(x, t) = f(x) v(t)$ to the wave equation initial boundary value problem

$$u_{tt} = a^2 u_{xx}$$
$$u(x, 0) = f(x)$$
$$u_t(x, 0) = 0$$

$u(x,t) = (\sin \omega x)\, v(t)$

$u_{tt} = (\sin \omega x)\, v''(t)$

$u_{xx} = -\omega^2 (\sin \omega x)\, v(t)$

$u_{tt} = a^2 u_{xx}$

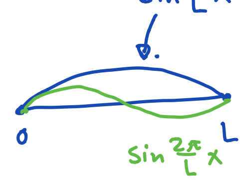
$(\cancel{\sin \omega x})\, v'' = a^2(-\omega^2)(\cancel{\sin \omega x})\, v$

$\begin{cases} v''(t) = -a^2\omega^2 v(t) \\ v(0) = 1, \; v'(0) = 0 \end{cases}$

$v''(t) + (\underbrace{a^2\omega^2}_{\text{const}})\, v(t) = 0$

$v(t) = c_1 \cos(a\omega t) + c_2 \sin(a\omega t)$

$v(0) = 1$, $v'(0) = 0$ $\Rightarrow c_1 = 1,\; c_2 = 0$



soltn $u = \sin\frac{\pi}{L}x \cos\left(\frac{\pi}{L} a t\right)$

soltn $u(x,t) = \sin\frac{2\pi}{L}x \cos\left(\frac{2\pi}{L} a t\right)$

vibrates twice as fast

If you also wanted to satisfy $u(0, t) = u(L, t) = 0$ for $t > 0$, what ω's could you use, and what is the list of product function solutions you would create?

so $v(t) = \cos(a\omega t)$

$u(x,t) = (\sin \omega x)(\cos(a\omega t))$

$\sin(\omega 0) = 0$ ✓

$\sin(\omega L) = 0$

$\omega = \frac{\pi}{L}, \frac{2\pi}{L}, \ldots \frac{n\pi}{L}$

$\sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{a\pi}{L}t\right)$

$\sin\left(\frac{2\pi}{L}x\right)\cos\left(a\frac{2\pi}{L}t\right)$

$\vdots$

$\sin\left(\frac{n\pi}{L}x\right)\cos\left(a\frac{n\pi}{L}t\right)$

If instead, you also wanted to satisfy $u_x(0, t) = u_x(L, t)$ for $t > 0$, what is the list of product function solutions you could create?

$\cos\left(\frac{\pi}{L}x\right)\cos\left(\frac{a\pi}{L}t\right)$

$\cos\left(\frac{2\pi}{L}x\right)\cos\left(\frac{2a\pi}{L}t\right)$

$\vdots$

$\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}at\right)$

$u_{tt} = a^2 u_{xx}$

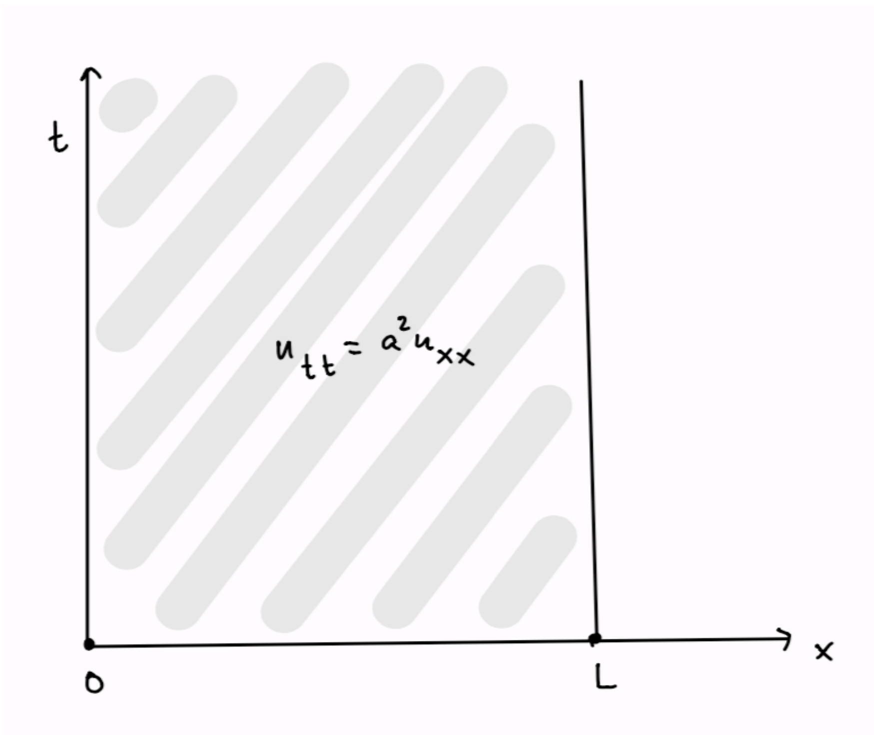
$\frac{\partial}{\partial x}\left(\cos\frac{n\pi}{L}x\right)\left(\cos\frac{n\pi}{L}at\right)$

$= -\frac{n\pi}{L}\left(\sin\frac{n\pi}{L}x\right)\left(\cos\frac{n\pi}{L}at\right)$

@ $x = 0, L$ this $\equiv 0$

$\left(\begin{matrix}\sin n\pi = 0 \\ \sin 0 = 0\end{matrix}\right)$

The natural initial value boundary value problems for the wave equation on finite-length intervals:



$$u_{tt} = a^2 u_{xx} \qquad 0 < x < L,\ t > 0$$

$$u(x,0) = f(x) \qquad \text{initial displacement}$$

$$u_t(x,0) = \cancel{g(x)}\ 0 \qquad \text{initial velocity}$$

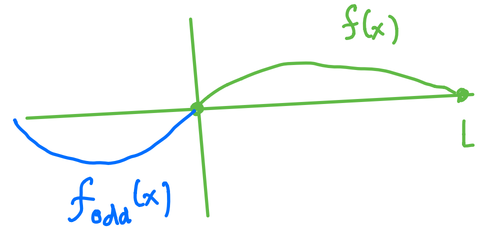
with either fixed endpoint or free endpoint boundary conditions:

type 1: fixed endpoints

$$u(0,t) = u(L,t) = 0 \qquad t > 0$$

type 2: free endpoints

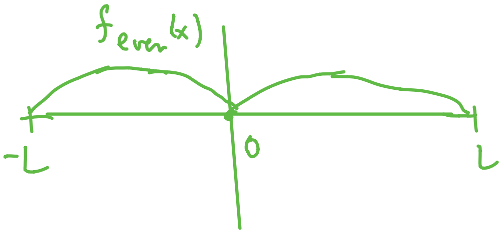
$$u_x(0) = u_x(L) = 0 \qquad t > 0$$



type 1: Odd extension of f

$$f_{odd}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}at\right)$$



type 2: even extension of f

$$f_{even}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}at\right)$$

Exercise 3

$$* \begin{cases} u_{tt} = a^2 u_{xx} & t>0,\ 0<x<L \\ u(x,0) = 0 \\ u_t(x,0) = g(x) \end{cases}$$

If you also wanted to satisfy $u(0, t) = u(L, t) = 0$ for $t > 0$, what ω's could you use, and what is the list of product function solutions you would create?

for $g(x) = \sin \omega x$ or $g(x) = \cos \omega x$

What product function $u(x,t) = g(x)v(t)$ solves $*$?

LHS $u_{tt} = (\cos \omega x) v''(t)$

RHS $a^2 u_{xx} = a^2(-\omega^2) \cos\omega x \, v(t)$

$=$ iff $v''(t) = -a^2\omega^2 v(t)$

$v''(t) + a^2\omega^2 v(t) = 0$

$v(t) = c_1 \cos(a\omega t) + c_2 \sin(a\omega t)$

$u(x,t) = (\cos\omega x)(c_1 \cos(a\omega t) + c_2 \sin(a\omega t))$

$u(x,0) = 0 \Rightarrow c_1 = 0$

$u_t(x,0) = \cos\omega x \Rightarrow c_2 = \frac{1}{a\omega}$

$\cos\omega x(-a\omega c_1 \sin(a\omega t) + a\omega c_2 \cos(a\omega t))$

@ $t=0$

$\cos\omega x (a\omega c_2)$

$$\left\{ \sin\left(\frac{n\pi}{L}x\right) \frac{1}{a\left(\frac{n\pi}{L}\right)} \sin\left(\frac{n\pi}{L}at\right) \right\}_{n\in\mathbb{N}}.$$

If instead, you also wanted to satisfy $u_x(0, t) = u_x(L, t)$ for $t > 0$, what is the list of product function solutions you could create?

after class

$$\left\{ \cos\left(\frac{n\pi}{L}x\right) \frac{L}{an\pi} \sin\left(\frac{n\pi}{L}at\right) \right\}_{n\in\mathbb{N}}$$

Use Fourier series for even and odd extensions of the initial value data below, to write down solutions to the *four* subtypes of initial boundary value problems

$$u_{tt} = a^2 u_{xx} \qquad 0 < x < L,\ t > 0$$

$$u(x,0) = \cancel{f(x)}\ 0 \qquad \text{initial displacement}$$

$$u_t(x,0) = g(x) \qquad \text{initial velocity}$$

with either fixed endpoint or free endpoint boundary conditions:

type 1: fixed endpoints

$$u(0,t) = u(L,t) = 0 \qquad t > 0$$

type 2: free endpoints

$$u_x(0) = u_x(L) = 0 \qquad t > 0$$

after class,

type 1 : odd 2L-periodic extension for g.

$$g_{odd}(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right).$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}at\right)\left(\frac{L}{n\pi a}\right)$$

type 2 : even 2L - periodic extension for g

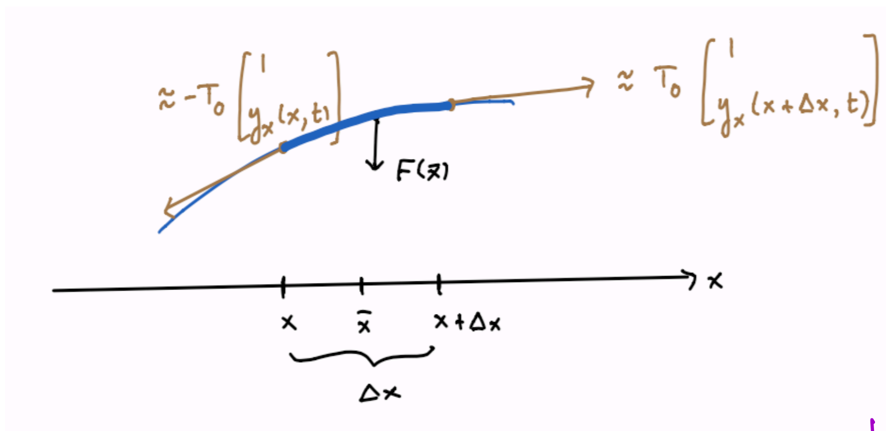
$$g_{even}(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L}at\right)\left(\frac{L}{n\pi a}\right)$$

Slinky waves:

The first thing you'll notice as we stretch the slinky horizontally is that gravity causes it to sag. Explain the sag shape as a nonhomogeneous particular solution to the transverse wave, by adding a force density from gravity, to our previous model that we discussed on Friday. Augment the notes below from that discussion:

Consider this force diagram. We apply Newton's law to approximate the transverse acceleration of the short segment. The transverse forces arise from the transverse components of the tension-induced forces at the endpoints. Let ρ_0 be the density (mass/length) of the spring.



gravity.

$$\rho_0(\Delta x)\, y_{tt} \approx T_0\big(y_x(x+\Delta x) - y_x(x)\big) \quad - g\rho_0 \Delta x$$

$$\rho_0\, y_{tt} \approx T_0 \frac{y_x(x+\Delta x) - y_x(x)}{\Delta x} \quad - g\rho_0$$

And in the limit as $\Delta x \to 0$,

$$y_{tt} = \frac{T_0}{\rho_0}\, y_{xx}. \quad - g$$

This is the wave equation for $y(x, t)$, with

$$y_{tt} = a^2\, y_{xx}\ , \quad a = \sqrt{\frac{T_0}{\rho_0}}\ .$$

<u>slinky Exercise 1</u> What are the particular solutions to the wave equation with gravity, which are constant in time? Which one satisfies $y(0, t) = y(L, t) = 0$?

constant solns

$$0 = y_{tt} = \frac{T_0}{\rho_0} y_{xx} - g$$

$$y_p(x) = \frac{g\rho_0}{T_0}(x)(x-L)$$

$$y_{xx} = \frac{g\rho_0}{T_0}$$

parabolas!

O.K., now focus on solutions to the homogeneous wave equation, i.e. after we subtract off the particular solution from Exercise 1.

Assume the equilibrium slinky has length ≈ 0, and is Hookes-like with mass m and constant k. So if it's stretched to length L the density and tension are given by

$$\rho = \frac{m}{L} \qquad T \approx k\,L.$$

If I'd weighed the spring and brought a tape measure we could've verified the Hookesian nature of the spring and computed k in addition to m to make quantitative predictions before running the experiments...

Transverse oscillations: The speed a of impulse waves is given by

$$a = \sqrt{\frac{T}{\rho}} = \sqrt{\frac{kL}{\frac{m}{L}}} = L\sqrt{\frac{k}{m}}\,.$$

Longitudinal (parallel) oscillations: The speed b of impules waves is given by

$$b = \sqrt{-T'(\rho)}\,.$$

Since

$$T = k\,L = k\frac{m}{\rho},$$

$$T'(\rho) = -\frac{k\,m}{\rho^2} = -\frac{k\,m}{\left(\frac{m}{L}\right)^2} = -\frac{k}{m}L^2$$

So

$$b = \sqrt{-T'(\rho)} = L\sqrt{\frac{k}{m}} = a.$$

And so the period of transverse and longitudinal waves with fixed endpoints should be

$$\frac{2\,L}{a} = 2\sqrt{\frac{m}{k}}\,.$$

<u>Exercise 2</u> Check that the speeds of transverse and longitudinal waves are the same in our slinky by timing their periods with fixed endpoints.

actually the longitudinal waves seemed to travel 5-10% faster. This is totally possible if the spring becomes non-Hooksian when L is large.

<u>Exercise 3</u> We will have noticed that transverse pulse waves come back upside (with our fixed endpoint condition). We can explain this by superposing a $2\,L$-periodic train of pulse waves moving one way, with a $2\,L$ period train of the opposite sign traveling in the opposite direction!

we didn't get to discuss this in detail yet..

Exercise 4 Alternate way to measure speed, using standing waves.   The solution to e.g. the initial boundary value problem with fixed endpoints

$$\begin{aligned} &u_{tt} = a^2 u_{xx} && 0 < x < L, t > 0 \\ &u(x, 0) = f(x) && 0 < x < L \\ &u_t(x, 0) = 0 && 0 < x < L \\ &u(0, t) = u(L, t) = 0 && t > 0 \end{aligned}$$

in terms of the odd extension Fourier series for f is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} x\right) \cos\left(n \frac{\pi}{L} a t\right).$$

So any solution which contains the fundamental mode $\sin\left(\frac{\pi}{L} x\right)\cos\left(\frac{\pi}{L} a\, t\right)$ has the fundamental period

$$T = \frac{2L}{a} = \frac{2L}{L\sqrt{\frac{k}{m}}} = 2\sqrt{\frac{m}{k}}$$

which is the same as our traveling wave periods from Exercise 2. Check this.

we were off by 15% or so. Repeat experiment?