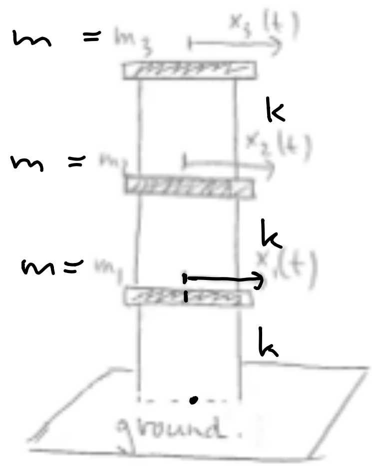
- An interesting shake-table demonstration:

  http://www.youtube.com/watch?v=M_x2jOKAhZM

Below is a discussion of how to model the unforced "three-story" building shown shaking in the video above, from which we can see which modes will be excited. There is also a "two-story" building model in the video, and its matrix and eigendata follow. Here's a schematic of the three-story building:



$$\cancel{m_3}\, x_3'' = -\frac{k}{m}(x_3 - x_2) = \frac{k}{m}(x_2 - x_3)$$

$$\cancel{m_2}\, x_2'' = -\frac{k}{m}(x_2 - x_1) + \frac{k}{m}(x_3 - x_2)$$

$$m_3\, x_1'' = -k\, x_1 + k(x_2 - x_1)$$

$\uparrow$ linearization constant

For the unforced (homogeneous) problem, the accelerations of the three massive floors (the top one is the roof) above ground and of mass m, are given by

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Note the -1 value in the last diagonal entry of the matrix. This is because $x_3(t)$ is measuring displacements for the top floor (roof), which has nothing above it. The "k" is just the linearization proportionality factor, and depends on the tension in the walls, and the height between floors, etc, as discussed on the previous page.

Exercise 4 Here is eigendata for the unscaled matrix $\left(\frac{k}{m}=1\right)$. For the scaled matrix you'd have the same eigenvectors, but the eigenvalues would all be multiplied by the scaling factor $\frac{k}{m}$ and the natural frequencies would all be scaled by $\sqrt{\frac{k}{m}}$ but the eigenvectors describing the modes would stay the same. Use this information describe the fundamental modes, and the order in which they will appear.

in phase sloshing

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = (c_1\cos\omega_3 t + c_2\sin\omega_3 t)\begin{bmatrix}.445\\.802\\1\end{bmatrix}$$

1st floor

3rd floor

$$+ (c_3\cos\omega_2 t + c_4\sin\omega_2 t)\begin{bmatrix}-1.25\\-.55\\1\end{bmatrix}$$

3rd floor out of phase w 1st two

$$+ (c_5\cos\omega_1 t + c_6\sin\omega_1 t)\begin{bmatrix}1.8\\-2.25\\1\end{bmatrix}$$

1st & 3rd floors in phase
middle is out of phase
with them & has
largest amplitude

predict order of excitation
as ω increases

Input:

eigenvalues $\frac{k}{m}\begin{pmatrix}-2 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -1\end{pmatrix}$

Results:

$\lambda_1 \approx -3.24698$ $\quad \omega_1 = 1.802\sqrt{\frac{k}{m}}$

$\lambda_2 \approx -1.55496$ $\quad \omega_2 = 1.247\sqrt{\frac{k}{m}}$ (4)

$\lambda_3 \approx -0.198062$ $\quad \omega_3 = .445\sqrt{\frac{k}{m}}$ (1)

Corresponding eigenvectors:

$v_1 \approx (1.80194, -2.24698, 1)$

$v_2 \approx (-1.24698, -0.554958, 1)$

$v_3 \approx (0.445042, 0.801938, 1)$

Input:

eigenvalues $\begin{pmatrix}-2 & 1\\ 1 & -1\end{pmatrix}$

Results:

$\lambda_1 \approx -2.61803$ $\quad \omega_1 = 1.618\sqrt{\frac{k}{m}}$ (5)

$\lambda_2 \approx -0.381966$ $\quad \omega_2 = .618\sqrt{\frac{k}{m}}$ (2)

Corresponding eigenvectors:

$v_1 \approx (-1.61803, 1)$

$v_2 \approx (0.618034, 1)$

single story (3)

$\omega = \sqrt{\frac{k}{m}}$

Tues April 2
5.6 Matrix exponentials.

• We'll finish Monday's notes first. •

Announcements: w12.1 a) Compute e^{tA} for $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ directly from the power series

from next week's assignment

hint: $\cosh t = \frac{1}{2}(e^t + e^{-t}) = 1 + \frac{t^2}{2!} + \cdots + \frac{t^{2n}}{(2n)!} + \cdots$

$\sinh t = \frac{1}{2}(e^t - e^{-t}) = t + \frac{t^3}{3!} + \cdots + \frac{t^{(2n+1)}}{(2n+1)!} + \cdots$

Warm-up Exercise: What are the McLaurin series for

used for $e^{i\theta}$ → $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots$

$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots + (-1)^n \frac{t^{2n}}{(2n)!} + \cdots$

$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$

$\frac{1}{1-t}$

Matrix exponentials. If you want to get a sense of the breadth of their applications in pure and applied math, consult the Wikipedia page on this topic! It also has a lot of the basic facts that we'll go through and use...

In the next three classes we'll talk about how matrix exponentials e^{tA} can be used to solve *all* homogeneous and non-homogeneous first order systems of differential equations with constant coefficient matrices A.

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$

So I sort of lied when I told you earlier in the course that for higher order linear DE's and for first order systems of DE's, there weren't explicit formulas for the solutions. There *are* explicit formulas, as long as the coefficient matrix is constant. The formulas and method will look exactly like a matrix-vectorized version of the method for scalar first order linear differential equation solutions that we studied in Chapter 1, that we solve with exponential integrating factor - namely solutions to

$$x'(t) - a\,x = f(t).$$

Definitions and properties:

Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Then

$$e^{A} := I + A + \frac{1}{2!}A^2 + \ldots + \frac{1}{n!}A^n + \ldots$$
$$\left(\Rightarrow e^{tA} := I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^n}{n!}A^n + \ldots \right)$$

(1) Note: the infinite sum converges: Let M be the maximum of all of the absolute values of the entries a_{ij}. Then the maximum absolute value of any entry in A^2 is at most $M^2 + M^2 + \ldots + M^2 = nM^2$. So the maximum absolute value of entry in A^3 is at most n^2M^3, etc; the maximum absolute value of any entry of A^m is at most $n^{m-1}M^m$.

$$\left|entry_{ij}\, e^{A}\right| \le 1 + M + \frac{1}{2!}nM^2 + \frac{1}{3!}n^2M^3 + \ldots$$
$$\le 1 + nM + \frac{1}{2!}(nM)^2 + \ldots + \frac{1}{n!}(nM)^n + \ldots = e^{nM} < \infty.$$

Since the series for each entry is absolutely convergent, it is also convergent. So the entries of the limit matrix exist and are numbers with absolute value less than e^{nM}.

(2) If A and B commute, i.e. $AB = BA$, then

$$e^{A}e^{B} = e^{B}e^{A}.$$

? "binomial theorem"

check: $\left(I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots\right)\left(I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \ldots\right)$

$= I + (A+B) + \frac{1}{2!}\left(A^2 + 2AB + B^2\right) + \frac{1}{3!}\left(A^3 + 3A^2B + 3AB^2 + B^3\right)$

e^Be^A would give analogous formula, except B'A's swapped.

If $AB = BA$ then the formulas

give same result

$$e^{A+B} = I + (A+B) + \frac{1}{2!}\underbrace{(A+B)^2}_{(A+B)(A+B)} + \frac{1}{3!}\overset{\text{"}}{(A+B)^3} + \cdots$$

$$\underbrace{(A+B)(A+B)}_{A^2 + AB + BA + B^2}$$

$= A^2 + 2AB + B^2$ if $AB = BA$

(3) If A and B commute, then

$$e^{A+B} = e^A e^B \qquad (= e^B e^A)$$

$(A+B)(A+B)(A+B)$

$= A^3 + \underbrace{A^2B + ABA + BA^2}_{3A^2B \text{ if } AB=BA}$

$+ \ldots$

$$e^{[0]} = I + [0] + \frac{1}{2!}[0]^2 + \ldots$$

$= I$

(4) So

$$e^A e^{-A} = e^{A-A} = e^{[0]} = I.$$

In other words, e^A is always invertible, and its inverse matrix is e^{-A}.

we'll see, the answer relates to syst.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Exercise 1 Use the power series definition to compute e^{tA} for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$e^{tA} = I + t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{t^3}{3!}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} +$$

$$+ \frac{t^4}{4!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{t^5}{5!}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \ldots$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$A^3 = AA^2 = -A$$

$$A^4 = AA^3 = A(-A) = -A^2 = I$$

$$= \begin{bmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots & t - \frac{t^3}{3!} + - \\ & \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$