Fri April 19
9.6 Introduction to the wave equation !!

Announcements:

- I discovered that Desmos makes movies
  ... if only I'd known before making the homework assignment
- We'll do some "slinky" math on Monday.

Warm-up Exercise: nope

from Wikipedia ...

Wave equation

From Wikipedia, the free encyclopedia

Not to be confused with Wave function.

The **wave equation** is an important second-order linear partial differential equation for the description of waves—as they occur in classical physics—such as mechanical waves (e.g. water waves, sound waves and seismic waves) or light waves. It arises in fields like acoustics, electromagnetics, and fluid dynamics.

Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli, and Joseph-Louis Lagrange.[1][2][3][4] In 1746, d'Alembert discovered the one-dimensional wave equation, and within ten years Euler discovered the three-dimensional wave equation.[5]

Introduction [edit]

The wave equation is a partial differential equation that may constrain some scalar function $u = u(x_1, x_2, \ldots, x_n; t)$ of a time variable t and one or more spatial variables $x_1, x_2, \ldots x_n$. The quantity u may be, for example, the pressure in a liquid or gas, or the displacement, along some specific direction, of the particles of a vibrating solid away from their resting positions. The equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \cdots + \frac{\partial^2 u_n}{\partial x_n^2}\right)$$

where c is a fixed non-negative real coefficient.

Using the notations of Newtonian mechanics and vector calculus, the wave equation can be written more compactly as

$$\ddot{u} = c^2 \nabla^2 u$$

where $\ddot{\square}$ denotes double time derivative, ∇ is the nabla operator, and $\nabla^2 = \nabla \cdot \nabla$ is the (spatial) Laplacian operator:

$$\ddot{u} = \frac{\partial^2 u}{\partial t^2} \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right) \qquad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

A solution of this equation can be quite complicated, but it can be analyzed as a linear combination of simple solutions that are sinusoidal plane waves with various directions of propagation and wavelengths but all with the same propagation speed c. This analysis is possible because the wave equation is linear; so that any multiple of a solution is also a solution, and the sum of any two solutions is again a solution. This property is called the superposition principle in physics.

theme in 2280

The wave equation alone does not specify a physical solution; a unique solution is usually obtained by setting a problem with further conditions, such as initial conditions, which prescribe the amplitude and phase of the wave. Another important class of problems occurs in enclosed spaces specified by boundary conditions, for which the solutions represent standing waves, or harmonics, analogous to the harmonics of musical instruments.

The wave equation is the simplest example of a hyperbolic differential equation. It, and its modifications, play fundamental roles in continuum mechanics, quantum mechanics, plasma physics, general relativity, geophysics, and many other scientific and technical disciplines.

We'll discuss the 1 space dimension wave equation, which models vibrating strings or air columns (as in musical instruments). In this case the function $u(x, t)$ measures the displacement of the point at position x, $0 \le x \le L$, at time t. One can consider displacements transverse or parallel to the direction of the configuration, just as we did for the coupled mass-spring systems. The homogeneous wave equation is the partial differential equation

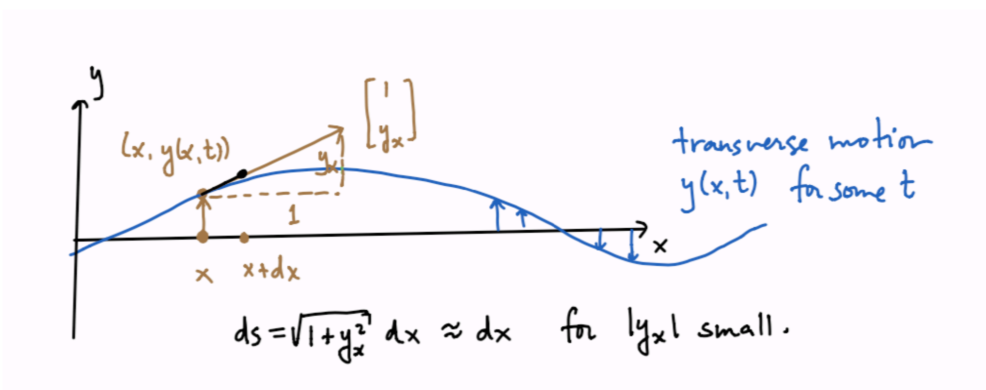
$$u_{tt} = a^2 u_{xx}$$

where $a > 0$ is a constant which turns out to be the speed at which waves travel. Notice that this is a linear homogeneous partial differential equation - the operator $L(u) := u_{tt} - a^2 u_{xx}$ is linear so that the solution space to the homogeneous wave equation is a vector space (linear combinations of solutions are solutions). The inhomogeneous wave equation is give by

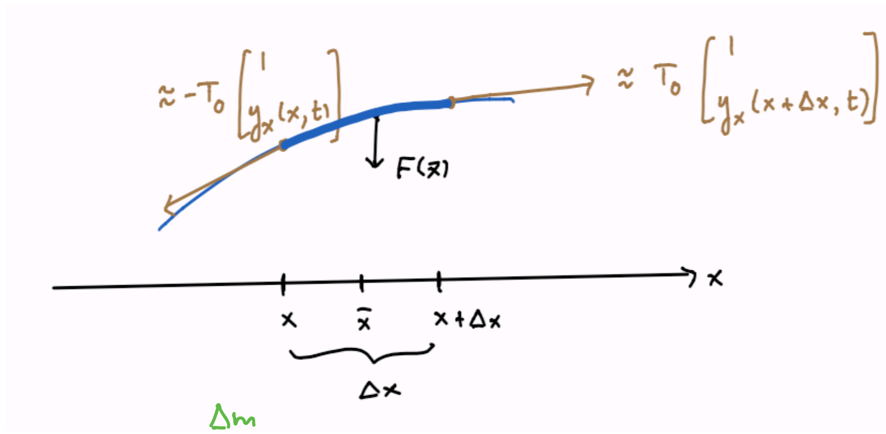
$$u_{tt} = a^2 u_{xx} + f(x, t)$$

where f represents external forces being applied at location x at time t.

Derivation: Let's first consider *transverse* waves along a stretched spring. In this case, in the linearized model, the tension is constant T_0 at equilibrium and remains constant as the spring vibrates because to first order the spring is not subject to additional stretching. Let's call the function we're studying $y(x, t)$ in this case:



Consider this force diagram. We apply Newton's law to approximate the transverse acceleration of the short segment. The transverse forces arise from the transverse components of the tension-induced forces at the endpoints. Let ρ_0 be the density (mass/length) of the spring.



$\frac{1}{\sqrt{1+y_x^2}} \approx 1$ linearize

$$\overbrace{\rho_0(\Delta x)}^{\Delta m}\, y_{tt} \approx T_0\big(y_x(x+\Delta x) - y_x(x)\big)$$ Newton's law

$$\rho_0\, y_{tt} \approx T_0 \frac{y_x(x+\Delta x) - y_x(x)}{\Delta x}$$

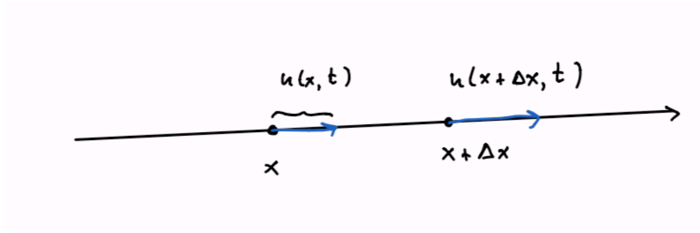
And in the limit as $\Delta x \to 0$,

$$y_{tt} = \frac{T_0}{\rho_0}\, y_{xx}.$$

This is the wave equation for $y(x, t)$, with

$$y_{tt} = a^2\, y_{xx}\,, \quad a = \sqrt{\frac{T_0}{\rho_0}}\,.$$

Parallel vibrations (also called compression waves). You can also consider vibrations in the direction of the spring. This also yields a wave equation. In this case you are stretching the spring and so the tension is not constant:



If the tension is expressed as a function of the spring density (which is changing depending on time and location), $T = T(\rho)$ then in this case the linearized wave equation turns out to be

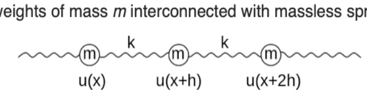
$$u_{tt} = b^2 u_{xx}, \qquad b^2 = -T'(\rho_0) > 0$$

We won't go through it in class, but Wikipedia has a derivation of the wave equation based on a limit of our favorite modeling for multi mass-spring systems using Hooke's law (shown below). It's also possible to derive the model analogously to how we worked out the transverse oscillation case, it's just a little messier than that case.

space dimension.[5]

From Hooke's law [edit]

The wave equation in the one-dimensional case can be derived from Hooke's Law in the following way: Imagine an array of little weights of mass *m* interconnected with massless springs of length *h*. The springs have a spring constant of *k*:



Here the dependent variable *u(x)* measures the distance from the equilibrium of the mass situated at *x*, so that *u(x)* essentially measures the magnitude of a disturbance (i.e. strain) that is traveling in an elastic material. The forces exerted on the mass *m* at the location *x* + *h* are:

$$F_{\text{Newton}} = m \cdot a(t) = m \cdot \frac{\partial^2}{\partial t^2} u(x+h,t)$$
$$F_{\text{Hooke}} = F_{x+2h} - F_x = k\left[u(x+2h,t) - u(x+h,t)\right] - k[u(x+h,t) - u(x,t)]$$

our favorite ☺

The equation of motion for the weight at the location *x* + *h* is given by equating these two forces:

$$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{k}{m}[u(x+2h,t) - u(x+h,t) - u(x+h,t) + u(x,t)]$$

where the time-dependence of *u(x)* has been made explicit.

If the array of weights consists of *N* weights spaced evenly over the length *L* = *Nh* of total mass *M* = *Nm*, and the total spring constant of the array *K* = *k*/*N* we can write the above equation as:

$$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{KL^2}{M} \frac{u(x+2h,t) - 2u(x+h,t) + u(x,t)}{h^2}.$$

Taking the limit *N* → ∞, *h* → 0 and assuming smoothness one gets:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{KL^2}{M} \frac{\partial^2 u(x,t)}{\partial x^2},$$

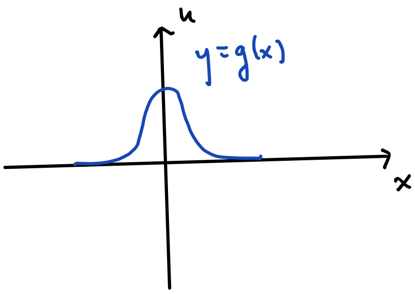
which is from the definition of a second derivative. KL^2/M is the square of the propagation speed in this particular case.

Special solutions There are two sorts of special solutions for the wave equation: product solutions $u(x,t) = X(x)T(t)$ analogous to the product solutions we used to solve the heat equation initial value problems. And, solutions of the following form, which indicate why we call this the "wave equation".

Exercise 1a For the wave equation

$$u_{tt} = a^2 u_{xx}$$

and for *any* twice continuously differentiable function $g(x)$, use the chain rule to show that $u_1(x,t) = g(x - a\,t)$ and $u_2(x,t) := g(x + a\,t)$ solve the wave equation.



$u(x) = g(x - at)$

$u_t = g'(x-at)(-a)$ Chain rule

$u_{tt} = g''(x-at)a^2$

$u_x = g'(x-at)\cdot 1$

$u_{xx} = g''(x-at)$

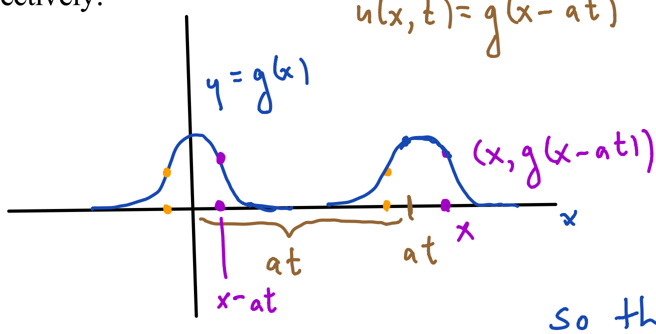
$u_{tt} = a^2 u_{xx}$?

$g''(x-at)a^2 = a^2 g''(x-at)$ ✓

also works for $u(x) = g(x+at)$

1b Interpret these solutions as waves traveling to the right with speed a, and to the left with speed a, respectively.

$u(x,t) = g(x-at)$



original profile, shifted to the right by at

so the profile of $u(x,t)$ is just the profile of $g(x)$ shifted by at

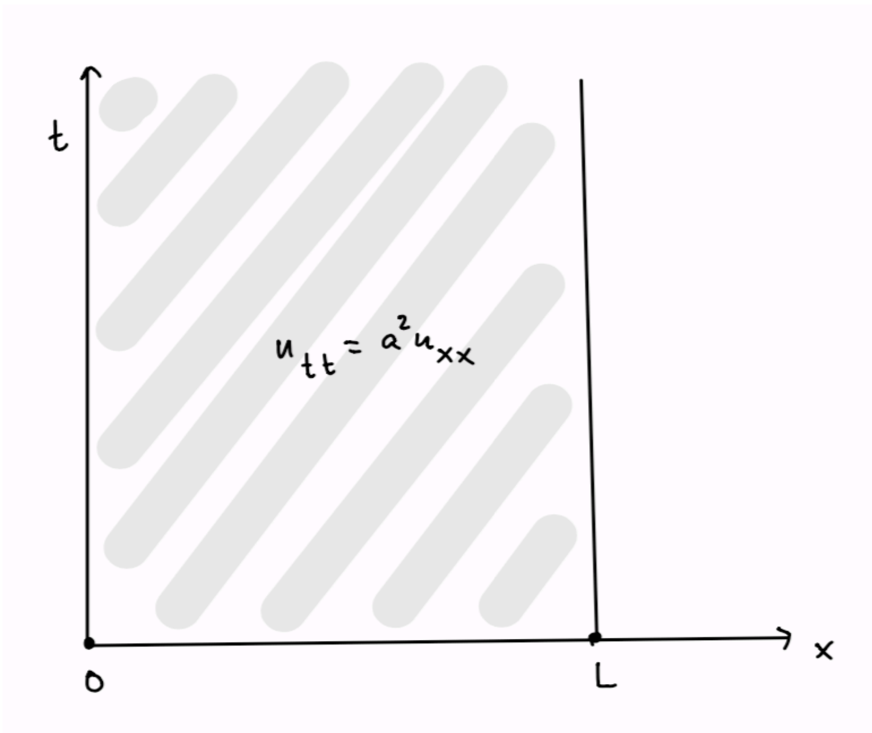
i.e moving with speed a

analogous for $u(x,t) = g(x+at)$, except wave moves to the left

1c For transverse vibrations, the speed $a = \sqrt{\dfrac{T_0}{\rho_0}}$. Does this comport with your intuition about how fast waves should travel as we vary the spring tension and density?

yes: higher tension → greater speed
higher density → slower speed

The natural initial value boundary value problems for the wave equation on finite-length intervals:



$$u_{tt} = a^2 u_{xx} \qquad 0 < x < L,\ t > 0$$

$$u(x, 0) = f(x) \qquad \text{initial displacement}$$

$$u_t(x, 0) = g(x) \qquad \text{initial velocity}$$

with either fixed endpoint or free endpoint boundary conditions:

type 1: fixed endpoints

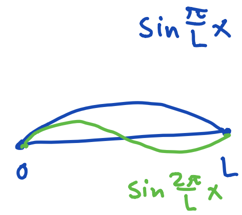
$$u(0, t) = u(L, t) = 0 \qquad t > 0$$

type 2: free endpoints

$$u_x(0) = u_x(L) = 0 \qquad t > 0$$

On Monday we'll show how to use Fourier series with superposition of product solutions to solve the natural initial value problems, analogously to how we used them for the heat equation. We'll also relate the Fourier series method to the traveling wave method we discussed in Exercise 1, and play with a slinky. If we have time today, we can do the following warm-up exercise:

<u>Exercise 2</u> Let $f(x) = \sin(\omega x)$ (or $f(x) = \cos(\omega x)$). Find a product solution $u(x, t) = f(x)v(t)$ to the wave equation initial boundary value problem



$$u_{tt} = a^2 u_{xx}$$
$$u(x, 0) = f(x)$$
$$u_t(x, 0) = 0$$

$u(x,t) = (\sin \omega x)\, v(t)$

$u_{tt} = (\sin \omega x)\, v''(t)$

$u_{xx} = -\omega^2 (\sin \omega x)\, v(t)$

$u_{tt} = a^2 u_{xx}$

$(\cancel{\sin \omega x})\, v'' = a^2(-\omega^2)(\cancel{\sin \omega x})\, v$

$\begin{cases} v''(t) = -a^2\omega^2 v(t) \\ v(0) = 1, \ v'(0) = 0 \end{cases}$

$v''(t) + (\underbrace{a^2\omega^2}_{\text{const}})\, v(t) = 0$

$v(t) = c_1 \cos(a\omega t) + c_2 \sin(a\omega t)$

$v(0) = 1$
$v'(0) = 0$ $\Rightarrow c_1 = 1, \ c_2 = 0$

If you also wanted to satisfy $u(0, t) = u(L, t) = 0$ for $t > 0$, what ω's could you use, and what is the list of product function solutions you would create?

so $v(t) = \cos(a\omega t)$

$u(x,t) = (\sin \omega x)(\cos(a\omega t))$

$\sin(\omega 0) = 0$ ✓

$\sin(\omega L) = 0$

$\omega = \frac{\pi}{L}, \frac{2\pi}{L}, \ldots \frac{n\pi}{L}$

$\sin\left(\frac{\pi}{L} x\right) \cos\left(\frac{a\pi}{L} t\right)$

$\sin\left(\frac{2\pi}{L} x\right) \cos\left(a\frac{2\pi}{L} t\right)$

$\vdots$

$\sin\left(\frac{n\pi}{L} x\right) \cos\left(a\frac{n\pi}{L} t\right)$

If instead, you also wanted to satisfy $u_x(0, t) = u_x(L, t)$ for $t > 0$, what is the list of product function solutions you could create?

Exercise 3 Let $g(x) = \sin(\omega x)$ (or $g(x) = \cos(\omega x)$). Find a product solution $u(x, t) = g(x)v(t)$ to the wave equation initial boundary value problem

$$u_{tt} = a^2 u_{xx}$$
$$u(x, 0) = 0$$
$$u_t(x, 0) = g(x)$$

If you also wanted to satisfy $u(0, t) = u(L, t) = 0$ for $t > 0$, what ω's could you use, and what is the list of product function solutions you would create?

If instead, you also wanted to satisfy $u_x(0, t) = u_x(L, t)$ for $t > 0$, what is the list of product function solutions you could create?