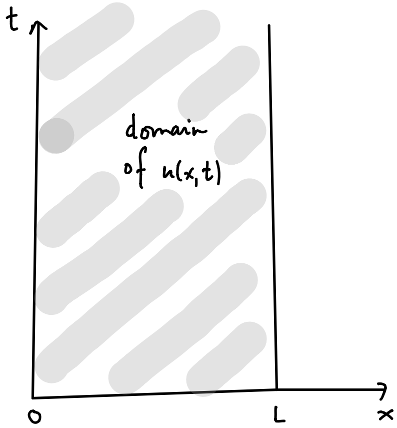
Two of the most often studied Initial boundary value problems (IBVP's)

type ①

$$\begin{cases} u_t = k u_{xx} & 0<x<L, \quad 0<t<\infty \\ u(x,0) = f(x) & \text{initial temp } 0<x<L \\ u(0,t) = 0 & t>0 \\ u(L,t) = 0 & t>0 \end{cases}$$

boundary temp. held constant (need not always be zero)



type ②

$$\begin{cases} u_t = k u_{xx} & 0<x<L, \quad 0<t<\infty \\ u(x,0) = g(x) & 0<x<L \\ u_x(0,t) = 0 & t>0 \\ u_x(L,t) = 0 & t>0 \end{cases}$$

no heat flux thru boundary (insulated end condition)

Example ① $\boxed{f(x) = \sin\left(\frac{\pi}{L}x\right)}$ or $\boxed{\sin\left(\frac{n\pi}{L}x\right)}$

product sol'n: try

$u(x,t) = v(t)\sin\omega x \qquad v(0)=1$

in the PDE

$u_t = k u_{xx}$

$v_t \cancel{\sin\omega x} = k v(-\omega^2 \cancel{\sin\omega x})$

$$\begin{cases} v_t = -k\omega^2 v \\ v(0) = 1 \end{cases}$$

$\Rightarrow v(t) = e^{-k\omega^2 t}$

find $\boxed{u(x,t) = \sin\left(\frac{\pi}{L}x\right)e^{-k\left(\frac{\pi}{L}\right)^2 t}}$

Solves ①. Also $\boxed{u(x,t) = \sin\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$

general solution to ①: Use sine series for f and superpose:

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

② $g(x) = \cos\left(\frac{n\pi}{L}x\right)$

$u(x,t) = \cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

general solution to ②

use cosine series for g and superpose the resulting product solutions!

$$g \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

the $-k$ is missing in your notes

type 1 example (boundary temperature forced to be zero), with $u(x,0) = 100,\ 0 < x < L,$ using a square wave odd extension sine series for the initial temperature:

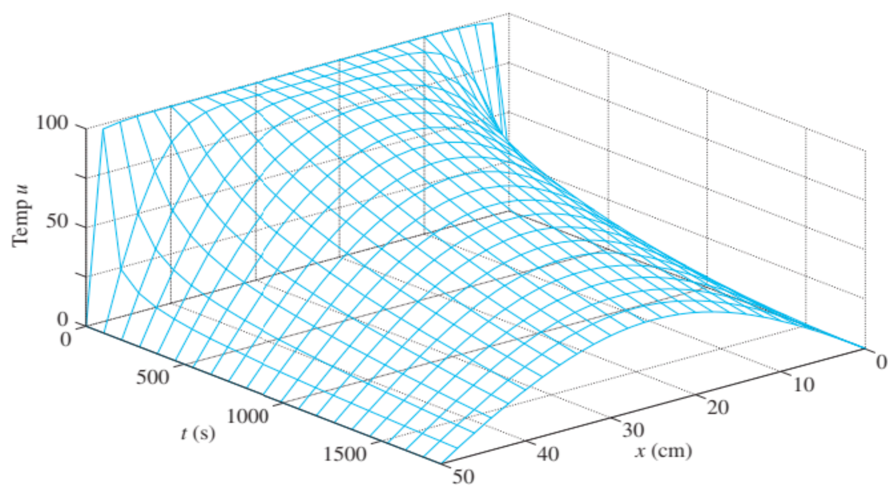


FIGURE 9.5.4. The graph of the temperature function $u(x,t)$ in Example 2.

type 2 example (zero heat flux through ends), with tent function inital data, and a tent function even extension cosine series. We'll do some computations for some examples tomorrow....

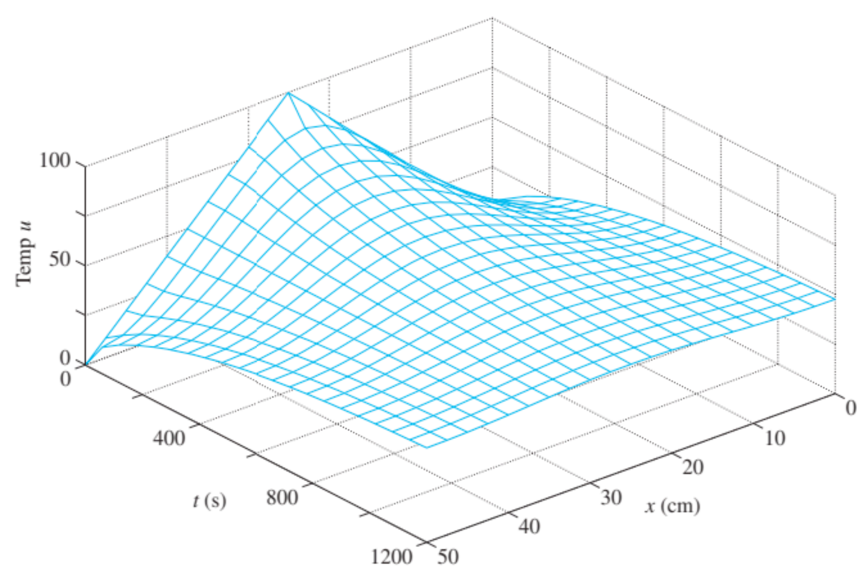


FIGURE 9.5.6. The graph of the temperature function $u(x,t)$ in Example 3.

Wed April 17
9.5 Solutions to initial boundary value problems for the heat equation.

Announcements: last quiz is today, hurray!
you'll find a Fourier series, and it won't be very difficult
(I hope).
Review next Fri a.m. or following Monday
(think about ...)

Warm-up Exercise: Suppose that the initial temperature of a rod of length $20\ cm$ is given by

$$f(x) = \sin\left(\frac{\pi}{20}x\right) \qquad 0 \le x \le 20$$

Find a solution $u(x, t) = f(x)v(t)$ to the initial boundary value problem

$$u_t = k\,u_{xx} \leftarrow \text{1 space dimension heat eqtn}$$
$$u(x, 0) = f(x)$$
$$u(0, t) = u(L, t) = 0.$$

$u(x,t)$ = temp @ time t
location x
k = thermal diffusivity

Hint: plug $u(x, t) = f(x)v(t)$ into the heat equation and deduce what $v(t)$ must be.

$$u(x,t) = \left(\sin\tfrac{\pi}{20}x\right)v(t) \qquad u(x,0) = f(x) = \sin\tfrac{\pi}{20}x$$

$$\Rightarrow u_t = \left(\sin\tfrac{\pi}{20}x\right)v'(t)$$
$$k\,u_{xx} = -\left(\tfrac{\pi}{20}\right)^2\left(\sin\tfrac{\pi}{20}x\right)v(t)$$

$$u_t = ku_{xx} \text{ iff } \begin{cases} v'(t) = -\left(\tfrac{\pi}{20}\right)^2 v(t)\,k \\ v(0) = 1 \end{cases}$$

Chapter 1 $\Rightarrow v(t) = e^{-\left(\frac{\pi}{20}\right)^2 t\,k}$

$v' = cv$
$v(0) = v_0$
$v(t) = v_0 e^{ct}$

$$u(x,t) = \left(\sin\tfrac{\pi}{20}x\right)e^{-\left(\frac{\pi}{20}\right)^2 t\,k}$$

decaying exponentially to zero

After finishing the discussion in Tuesday's notes, let's work one of the text examples carefully. We may not have time for both before the quiz. They illustrate the fixed endpoint temperature and zero heat flux boundary conditions, and are both interesting.

<u>Example 2</u>

Example 2 Suppose that a rod of length $L = 50$ cm is immersed in steam until its temperature is $u_0 = 100°C$ throughout. At time $t = 0$, its lateral surface is insulated and its two ends are imbedded in ice at 0°C. Calculate the rod's temperature at its midpoint after half an hour if it is made of (a) iron; (b) concrete.

$u(x,0) = 100, \quad 0<x<50$

Hint: To get the Fourier series for the initial temperature distribution consistent with the zero boundary values for $u(x,t)$ adapt the formula for the 2π-periodic square wave (now written as a function of x)

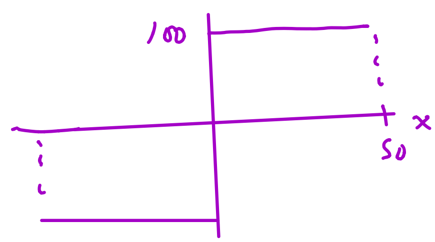
$$sq(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

$$sq(x) = \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n} \sin(n x)$$

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = 100 & 0<x<50 \\ u(0,t) = 0 & \\ u(50,t) = 0 & \end{cases} \quad t>0$$

And then use superposition of solutions:

$$f(x) = \sin\left(\frac{n\pi}{L}x\right) \qquad u(x,t) = \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$



consider

$-50<x<50$: $\quad sq\left(\frac{\pi}{50}x\right) = \begin{cases} -1 & -50<x<0 \\ 1 & 0<x<50 \end{cases}$

odd extension,

$$f(x) = \begin{cases} 100 & 0<x<50 \\ -100 & -50<x<0 \end{cases}$$

we want $u(x,0) = 100\, sq\left(\frac{\pi}{50}x\right)$

$$u(x,0) = \frac{400}{\pi} \sum_{n\ odd} \frac{1}{n} \sin\left(\frac{n\pi}{50}x\right)$$

$$\boxed{u(x,t) = \frac{400}{\pi} \sum_{n\ odd} \frac{1}{n} \sin\left(\frac{n\pi}{50}x\right) e^{-k\left(\frac{n\pi}{50}\right)^2 t}}$$

Material	k (cm^2/s)
Silver	1.70
Copper	1.15
Aluminum	0.85
Iron	0.15
Concrete	0.005

FIGURE 9.5.3. Some thermal diffusivity constants.

and therefore the rod's temperature function is given by

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \exp\left(-\frac{n^2\pi^2kt}{L^2}\right) \sin\frac{n\pi x}{L}.$$

$u_0 = 100$
$L = 50$

Figure 9.5.4 shows a graph $u = u(x,t)$ with $u_0 = 100$ and $L = 50$. As t increases, we see the maximum temperature of the rod (evidently at its midpoint) steadily decreasing. The temperature at the midpoint $x = 25$ after $t = 1800$ seconds is

$$u(25, 1800) = \frac{400}{\pi} \sum_{n \text{ odd}} \frac{(-1)^{n+1}}{n} \exp\left(-\frac{18n^2\pi^2k}{25}\right).$$

(a) With the value $k = 0.15$ that was used in Fig. 9.5.4, this series gives

$$u(25, 1800) \approx 43.8519 - 0.0029 + 0.0000 - \cdots \approx 43.85^\circ\text{C}.$$

This value $u(25, 1800) \approx 43.85$ is the maximum height (at its midpoint $x = 25$) of the vertical sectional curve $u = u(x, 1800)$ that we see at one "end" of the temperature surface shown in Fig. 9.5.4.
(b) With $k = 0.005$ for concrete, it gives

$$\begin{aligned} u(25, 1800) \approx\ & 122.8795 - 30.8257 + 10.4754 - 3.1894 \\ & + 0.7958 - 0.1572 + 0.0242 - 0.0029 \\ & + 0.0003 - 0.0000 + \cdots \approx 100.00^\circ\text{C}. \end{aligned}$$

Thus concrete is a very effective insulator. ■

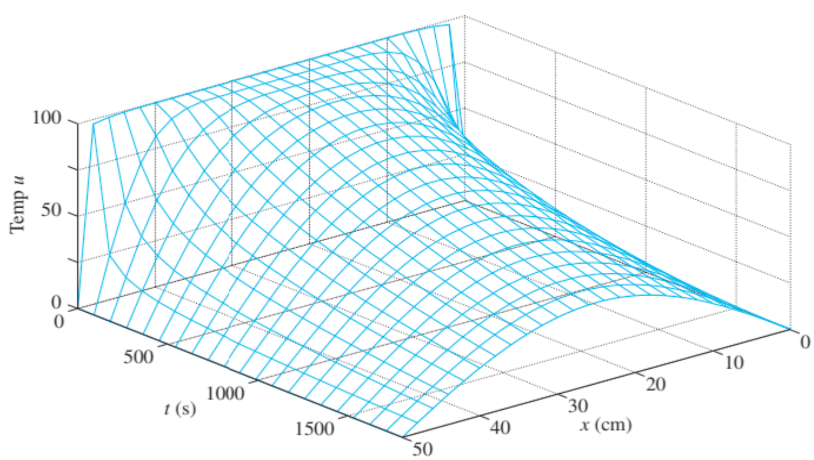


FIGURE 9.5.4. The graph of the temperature function $u(x,t)$ in Example 2.

Example 3

We consider the same 50-cm rod as in Example 2, but now suppose that its initial temperature is given by the "triangular function" graphed in Fig. 9.5.5. At time $t = 0$, the rod's lateral surface *and* its two ends are insulated. Then its temperature function $u(x, t)$ satisfies the boundary value problem

$$u_t = k u_{xx},$$
$$u_x(0, t) = u_x(50, t) = 0,$$
$$u(x, 0) = f(x).$$

Example 3

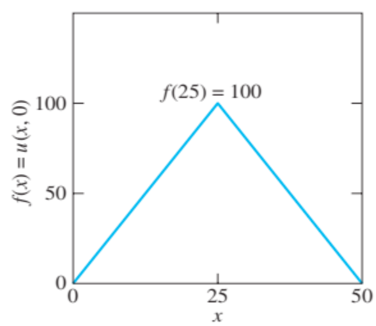


FIGURE 9.5.5. The graph of the initial temperature function $u(x, 0) = f(x)$ in Example 3.

Hint: adapt the Fourier series for the 2π - periodic tent function in order to express the initial temperature distribution as a *cosine* series (for the zero-flux boundary condition)

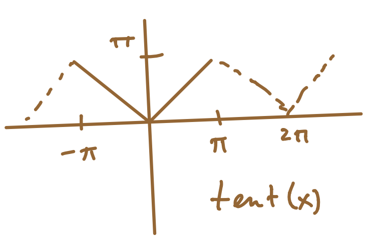
$$tent(x) \quad 2\pi\text{ - periodic}$$
$$tent(x) = |x|, \ -\pi \le x \le \pi.$$
$$tent(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n^2} \cos(n x)$$

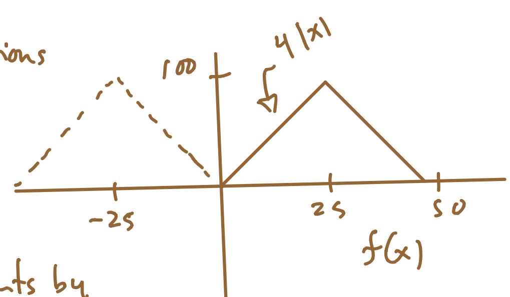
And then use superposition of solutions:

$$g(x) = \cos\left(\frac{n\pi}{L} x\right) \qquad u(x, t) = \cos\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

filled in after class:

even functions





scale inputs by $\frac{\pi}{25}$ to use tent:

$$tent\left(\frac{\pi}{25}x\right) = \left|\frac{\pi}{25}x\right| = \frac{\pi}{25}|x| \quad , \quad -25 < x < 25$$

$$\Rightarrow f(x) = 4|x| = \frac{100}{\pi}\, tent\left(\frac{\pi}{25}x\right) \qquad -25 < x < 25$$

$$\Rightarrow f(x) = \frac{100}{\pi}\left(\frac{\pi}{2} - \frac{4}{\pi}\right) \sum_{n\ odd} \frac{1}{n^2} \cos\left(\frac{n\pi}{25}x\right)$$

$$f(x) = 50 - \frac{400}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos\left(\frac{n\pi}{25} x\right)$$

for initial temperature data $\cos\left(\frac{n\pi}{25}x\right)$

get heat eqtn solution $\cos\left(\frac{n\pi}{25}x\right) e^{-k\left(\frac{n\pi}{25}\right)^2 t}$

the constant function 50 solves $u_t = k u_{xx}$

So!

$$\boxed{u(x,t) = 50 - \frac{400}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos\left(\frac{n\pi}{25}x\right) e^{-k\left(\frac{n\pi}{25}\right)^2 t}}$$

the text (next page) indexes the sum in a strange way, but gives an equivalent solution formula

$$u(x,t) = 50 - \frac{1600}{\pi^2} \sum_{n=2,6,10,\ldots} \frac{1}{n^2} \cos\left(\frac{n\pi}{50} x\right) e^{-k\left(\frac{n\pi}{50}\right)^2 t}$$

(their "n's" are twice ours)

$$u(x,t) = 50 - \frac{1600}{n^2} \sum_{n=2,6,10,\ldots} \frac{1}{n^2} \exp\left(-\frac{n^2\pi^2 kt}{2500}\right) \cos\frac{n\pi x}{50}.$$

Figure 9.5.6 shows the graph $u = u(x,t)$ for the first 1200 seconds, and we see the temperature in the rod beginning with a sharp maximum at the midpoint $x = 25$, but rapidly "averaging out" as the heat in the rod is redistributed with increasing t. ■

Finally, we point out that, although we set up the boundary value problems in (12) and (32) for a rod of length L, they also model the temperature $u(x,t)$ within the infinite slab $0 \leqq x \leqq L$ in three-dimensional space if its initial temperature $f(x)$ depends only on x and its two faces $x = 0$ and $x = L$ are either both insulated or both held at temperature zero.

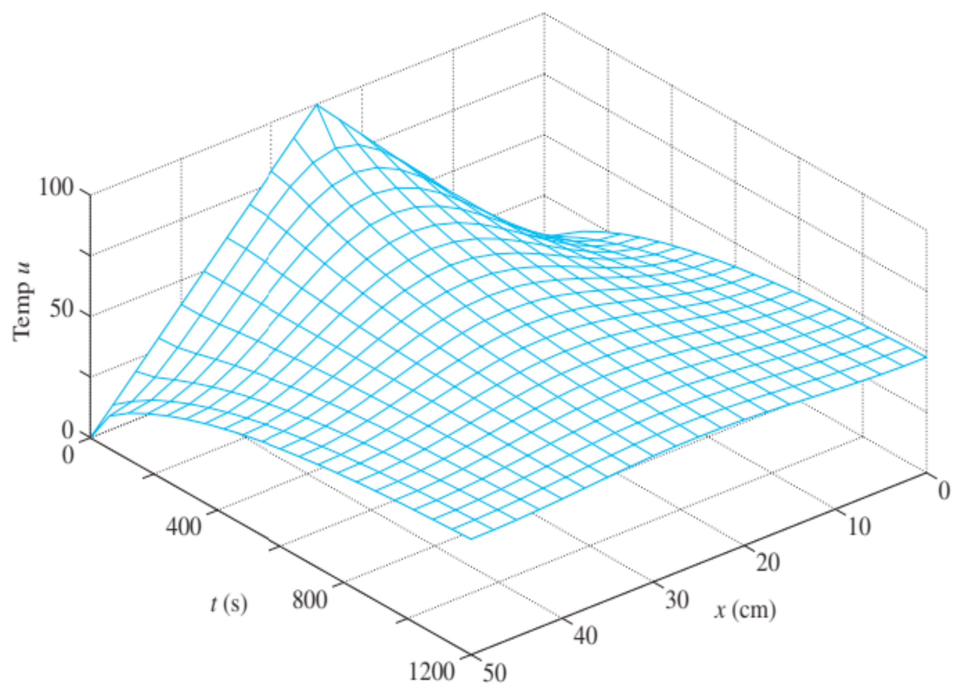


FIGURE 9.5.6. The graph of the temperature function $u(x,t)$ in Example 3.