from Friday April 5:

Exercise 1 Using the matrix exponential solution formula for the system, and the correspondence back to the second order DE, verify that the solution to the original second order IVP for the swing is

$x'' + x = f$

$$x(t) = x_0 \cos(t) + v_0 \sin(t) + \int_0^t f(s)\sin(t-s)\,ds$$

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \; ; \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}$$

$$\vec{x}(t) = e^{tA}\vec{x}_0 + \int_0^t e^{(t-s)A}\vec{f}(s)\,ds$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \vec{x}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} + \int_0^t \begin{bmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{bmatrix}\begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds$$

$x(0) = x_0$
$x'(0) = v_0$

$$\longrightarrow \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = x_0\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + v_0\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \int_0^t \begin{bmatrix} f(s)\sin(t-s) \\ f(s)\cos(t-s) \end{bmatrix} ds$$

$$x(t) = x_0\cos t + v_0 \sin t + \int_0^t f(s)\sin(t-s)\,ds$$

Any cont. fcn !!

Example 2 from April 5:

$$x''(t) + x(t) = sq(t)$$

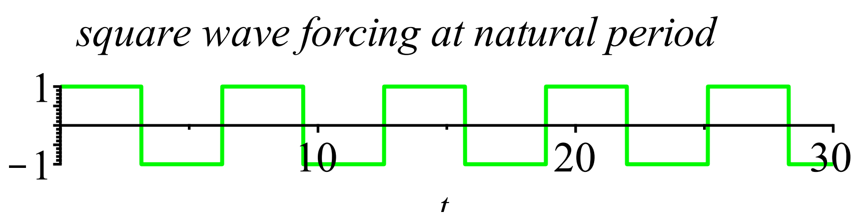
with

$$sq(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

and 2π - periodic. This forcing function appeared to cause resonance:

```
> with(plots) :
> f2 := t -> -1 + 2*(Sum((-1)^n*Heaviside(t - n*Pi), n = 0..10)) :
  plot2a := plot(f2(t), t = 0..30, color = green) :
  display(plot2a, title = `square wave forcing at natural period`);
```

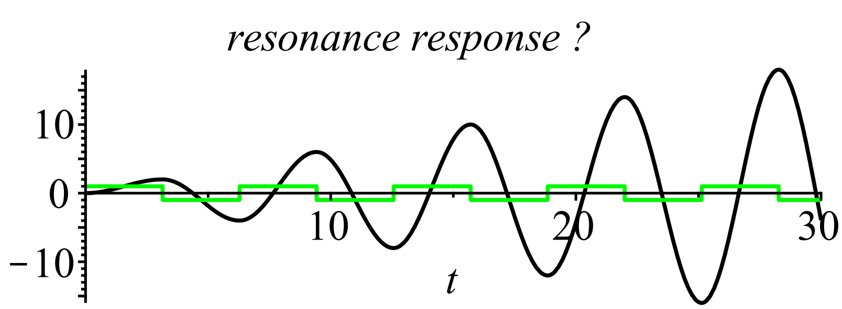
$$f2 := t \rightarrow -1 + 2\cdot\left(\sum_{n=0}^{10} (-1)^n \cdot \mathrm{Heaviside}(t - n\cdot \mathrm{Pi})\right):$$



What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

$$x2 := t \rightarrow \int_0^t \sin(\tau)\cdot f2(t-\tau)\, d\tau :$$

```
  plot2b := plot(x2(t), t = 0..30, color = black) :
  display({plot2a, plot2b}, title = `resonance response ?`);
```



<u>Exercise 2</u> Use the Fourier series for $sq(t)$ that we've found before

$$sq(t) = \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n} \sin(n\,t)$$

and infinite superposition to find a particular solution to

$$x''(t) + x(t) = sq(t)$$

that explains why resonance occurs. Make use of the undetermined coefficients particular solution formulas at the end of today's notes (and in the handout).

$$x''(t) + x = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \dots\right)$$

$$x(t) = c_1\cos t + c_2 \sin t$$

$$+ \frac{4}{\pi}\left(-\frac{t}{2}\cos t + \frac{1}{3}\frac{1}{1-9}\sin 3t + \dots\right)$$

$$x'' + \omega_0^2 x = A\sin\omega t$$

$$\omega = \omega_0: \quad x_p = -\frac{t}{2\omega_0} A\cos\omega_0 t$$

$$\omega \neq \omega_0: \quad x_p = \frac{A}{\omega_0^2 - \omega^2}\sin\omega t$$

$$= c_1\cos t + c_2\sin t - \frac{2t}{\pi}\cos t + \underbrace{\frac{4}{\pi}\sum_{\substack{n\ odd\\ n\geq 3}} \frac{1}{n}\underbrace{\frac{1}{1-n^2}}\sin nt}$$

$$|\sin nt| \leq 1$$

$$\left|\quad\right| \leq \frac{4}{\pi}\sum_{\substack{n\ odd\\ n\geq 3}}\left|\quad\right| \leq \frac{4}{\pi}\underbrace{\sum_{\substack{n\ odd\\ n\geq 3}} \frac{1}{n(n^2-1)}}$$

$$\leq \frac{4}{\pi}\sum_{all\ n\geq 3} \frac{1}{n^3}\left[\frac{n^2}{n^2-1}\right]$$

$n=3$: $\frac{9}{8}$ is largest for $n\geq 3$

$$\leq \frac{4}{\pi}\frac{9}{8}\underbrace{\sum_{n\geq 3}\frac{1}{n^3}}$$

$$\leq \frac{\cancel{4}}{\pi}\frac{9}{8}\frac{1}{\cancel{8}\,2}$$

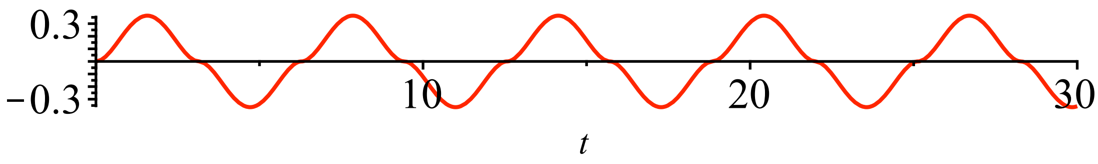
$$\leq \int_2^\infty \frac{1}{x^3}\,dx = \frac{x^{-2}}{-2}\Big]_2^\infty = \frac{1}{8}$$

$$\leq \frac{9}{16\pi} \approx .2$$

If we remove the $\sin(t)$ term from the square wave forcing function, and re-use the convolution formula, we see that we've eliminated the resonance, even though we're still forcing with a function that has the natural period.

$$> x2 := t \to \int_0^t \sin(\tau)\cdot\left(square(t-\tau) - \frac{4}{\pi}\cdot\sin(t-\tau)\right)\mathrm{d}\tau :$$

$plot(x2(t), t = 0..30);$



this is larger than our error estimate because $x_2(t)$ solves $x(0)=0$, $x'(0)=0$,

So although the c_1 of $x_H=0$, $c_2 \neq 0$. that's why a $c_2 \sin t$ dominates the graph above.

Exercise 3) Understand this example from April 5, using Fourier series

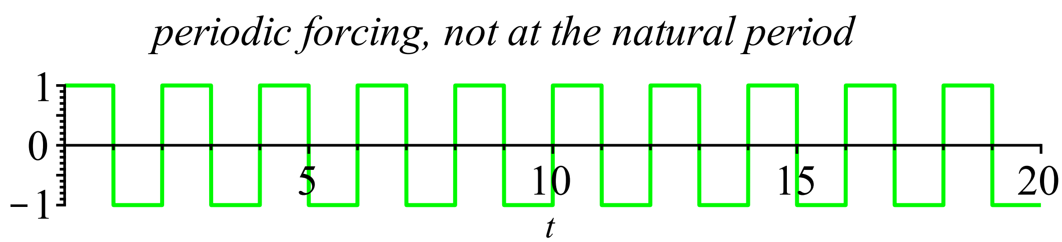
$$x''(t) + x(t) = f_4(t)$$

Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

$$f_4(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases}$$

```
> f4 := t -> -1 + 2·Sum((-1)^n·Heaviside(t - n), n = 0..20) :
  plot4a := plot(f4(t), t = 0..20, color = green) :
  display(plot4a, title = `periodic forcing, not at the natural period`);
```

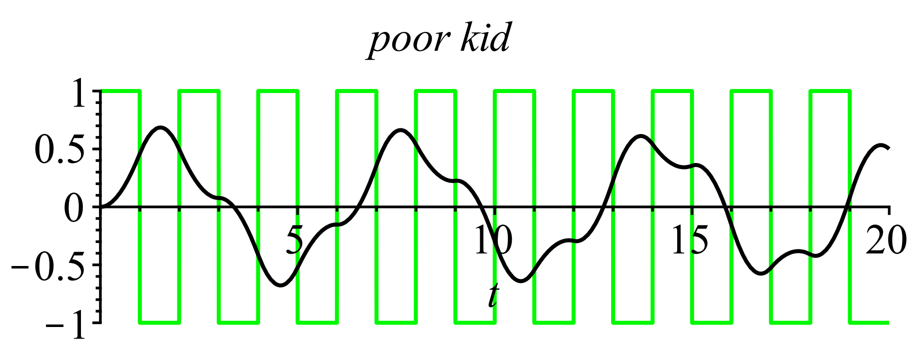
$$f4 := t \to -1 + 2\cdot\sum_{n=0}^{20} (-1)^n \cdot \mathrm{Heaviside}(t-n):$$



Resonance?

```
> x4 := t -> int(sin(τ)·f4(t - τ), τ = 0..t) :
  plot4b := plot(x4(t), t = 0..20, color = black) :
  display({plot4a, plot4b}, title = `poor kid `);
```

$$x4 := t \to \int_0^t \sin(\tau)\cdot f4(t-\tau)\, d\tau:$$



2π per. $Sq(t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin nt$

Hint: By rescaling we can express $f_4(t) = square(\pi\, t) = \frac{4}{\pi}\sum_{n\ odd}\frac{1}{n}\sin(n\,\pi\, t)$.

$$x'' + 1x = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin(n\pi t)$$

RHS	
$A\sin\omega t$ $\omega \neq \omega_0$	$x_p = \frac{A}{\omega_0^2 - \omega^2}\sin\omega t$

$$x(t) = c_1\cos t + c_2\sin t$$
$$+ \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\frac{1}{1^2 - (n\pi)^2}\sin(n\pi t)$$

check for absolute convergence.

$$x = c_1 \cos t + c_2 \sin t + \frac{4}{\pi} \frac{1}{1-\pi^2} \sin \pi t$$

$$+ \frac{4}{\pi} \sum_{\substack{n \geq 3 \\ n \text{ odd}}} -\frac{1}{n} \frac{1}{(n^2\pi^2-1)} \sin n\pi t$$

leading term (sign is wrong)

!

sign is correct!

... what we're actually seeing in the convolution solution graph is the $c_2 \sin t$ term for the IVP $x(0)=0$, $x'(0)=0$.

which has larger amplitude period 2π than the $\sin \pi t$ term.

abs convergence test

$$\frac{4}{\pi} \sum |\cdots|$$

$$\leq \frac{4}{\pi} \sum_{\substack{n \geq 3 \\ n \text{ odd}}} \frac{1}{n(n^2\pi^2-1)}$$

$$\leq \underbrace{\frac{4}{\pi} \sum_{n \geq 3} \frac{C}{n^3}}_{\text{small.}}$$

Practical resonance example:

Exercise 4 The steady periodic solution to the differential equation

$$x''(t) + .2\cdot x'(t) + 1\,x(t) = sq(t)$$

exhibits practical resonance. Explain this with Fourier series. Hint: Use

$$sq(t) = \frac{4}{\pi}\sum_{n\ odd}\frac{1}{n}\sin(n\,t)$$

and the table of particular solutions at the end of today's notes.

$$x'' + .2x' + x = \frac{4}{\pi}\sum_{n\ odd}\frac{1}{n}\sin nt = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \ldots\right)$$

$$x'' + cx' + \omega_0^2 x = A\sin\omega t$$

$$x_{sp} = C\sin(\omega t - \alpha)$$

$$C = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2 + c^2\omega^2}}$$

$$\alpha = \ldots$$

$$x = x_H + x_p$$

$x_H \to 0$, x_{tr}, $c>0$

$x_p \to x_{sp}$

$$x_p = \frac{4}{\pi}\frac{1}{\sqrt{(1-1)^2 + .04}}\sin(t-\alpha_1) + \frac{4}{\pi}\sum_{\substack{n\ odd\\ n\geq 3}}\frac{1}{n}\underbrace{\frac{1}{\sqrt{(1-n^2)^2 + .04n^2}}}_{\left|\ \right| \leq \frac{1}{n}\frac{1}{n^2-1}}\sin(nt-\alpha_n)$$

$$\frac{4}{\pi}5\sin(t-\alpha_1)$$

$$\left(5 = \frac{1}{.2}\right)$$

$6.5 \approx \frac{20}{\pi}$

"big"

$$\left(\sqrt{(1-n^2)^2} = n^2-1 \quad n\geq 3\right)$$

sum is tiny.

Particular solutions from Chapter 3:

$$x''(t) + \omega_0^2 x(t) = A\, sin(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2} \sin(\omega t) \quad \text{when } \omega \neq \omega_0$$

$$x_P(t) = -\frac{t}{2\,\omega_0} A \cos(\omega_0\, t) \quad \text{when } \omega = \omega_0$$

....

$$x''(t) + \omega_0^2 x(t) = A \cos(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t) \quad \text{when } \omega \neq \omega_0$$

$$x_P(t) = \frac{t}{2\,\omega_0} A \sin(\omega_0\, t) \quad \text{when } \omega = \omega_0$$

....

$$x'' + c\,x' + \omega_0^2 x = A \cos(\omega t) \qquad c > 0$$

$$x_P(t) = x_{sp}(t) = C \cos(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \quad .$$

$$\cos(\alpha) = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \quad .$$

....

$$x'' + c\,x' + \omega_0^2 x = A \sin(\omega t) \qquad c > 0$$

$$x_P(t) = x_{sp}(t) = C \sin(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \quad .$$

$$\cos(\alpha) = \frac{\omega^2 - \omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

Tues April 16
9.5 Introduction to the heat equation

Announcements: finish Monday's forced oscillations, 9.4, (~~15~~ 25 minutes)
start §9.5 (25 ")

T, W $\}$ heat eqtn §9.5

F, M $\}$ wave eqtn §9.6.

T $\}$ probably review.

HW 14 due Tues 4/23 @ 5:00 pm
(I'll accept it Wed. until noon)

Warm-up Exercise: nope.

From Wikipedia "heat equation"

Heat equation

From Wikipedia, the free encyclopedia

In physics and mathematics, the **heat equation** is a partial differential equation that describes how the distribution of some quantity (such as heat) evolves over time in a solid medium, as it spontaneously flows from places where it is higher towards places where it is lower. It is a special case of the diffusion equation.

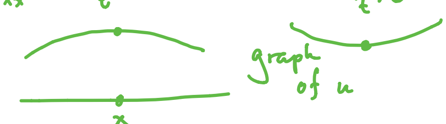
This equation was first developed and solved by Joseph Fourier in 1822 to describe heat flow. However, it is of fundamental importance in diverse scientific fields. In probability theory, the heat equation is connected with the study of random walks and Brownian motion, via the Fokker–Planck equation. In financial mathematics it is used to solve the Black–Scholes partial differential equation. A variant was also instrumental in the solution of the longstanding Poincaré conjecture of topology.

temp @ time t & location x

We'll focus on the one dimensional homogeneous heat equation for functions $u(x, t)$ where $0 \leq x \leq L$ is the one-dimensional spatial coordinate, for example along a metal bar of length L, and $t \geq 0$ is time. The heat equation partial differential equation reads as

$ku_{xx} = u_t < 0$ $\quad$ $u_t > 0$

$$L(u) := u_t - k u_{xx} = 0 \qquad u_t = k\, u_{xx}$$



in this case. The subscripts refer to partial derivatives, and the positive constant k is called the *diffusivity constant* or *thermal diffusivity*. The larger values of k are associated with quicker diffusions of heat from hot to cold regions, and depend on the material being studied:

Material	**k (cm^2/s)**
Silver	1.70
Copper	1.15
Aluminum	0.85
Iron	0.15
Concrete	0.005

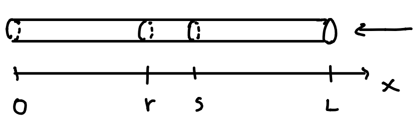
FIGURE 9.5.3. Some thermal diffusivity constants.

We won't study it, but the *nonhomogeneous heat equation* is given by

$$u_t = k\, u_{xx} + f(x, t)$$

where $f(x, t)$ represents a rate at which heat is being put into or taken out of the system, at location x and time t.

derivation



← cross-sectional area "a"

"Heat" is a measurement of energy

If $u(x,t)$ is absolute temp (° Kelvin) then the energy in the test interval $r \le x \le s$ is

$$E(t) = \int_r^s u(x,t)\, c\ \delta\ \underbrace{a\,dx}_{dV\ cm^3}$$

δ: density gm/cm^3; $\underbrace{\delta\ a\,dx}_{}$ = dm mass gm

c: specific heat calories/gm = energy required to heat 1 gm by 1° C

$c\,\delta a\,dx$ = calories = energy required to heat dx-piece 1°

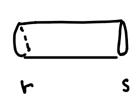
$u(x,t)\,c\,\delta a\,dx$ = energy in dx-piece

So $E(t)$ is total energy in the test interval

Assuming lateral insulation (so heat can only leave through the ends), the linear model is

$$\frac{dE}{dt} = Ka\left(u_x(s) - u_x(r)\right)$$

heat flow through ends proportional to temperature gradient u_x. Note the signs of u_x: heat flows from hot to cold (think vibrating molecules)



$$\frac{d}{dt}\int_r^s u(x,t)\, c\,\delta a\, dx = Ka\left(u_x(s) - u_x(r)\right)$$

$$\int_r^s \frac{\partial u}{\partial t}\, c\,\delta a\, dx = Ka\int_r^s u_{xx}\, dx$$

$\div \frac{1}{s-r}$, assume integrands are continuous functions, then take $\lim_{s\to r}$:

$$u_t\, c\,\delta \cancel{a} = K\cancel{a}\, u_{xx}$$

$$u_t = \frac{K}{c\delta}\, u_{xx} \qquad \boxed{u_t = k\, u_{xx}} \qquad k = \frac{K}{c\delta}$$

- If u satisfies $u_t = ku_{xx}$ then so does $v = c_1 u + c_2$ so you may use Celsius, Farenheit, Kelvin etc.

  since L is linear & "u" & "1" are homogeneous solutions

- $L(u) := u_t - ku_{xx}$ is a linear operator, so

$$u_t = ku_{xx}$$

is a linear homogeneous partial differential equation
– and linear combinations of solutions are solutions

(When people study the non-homogeneous problem

$$u_t = ku_{xx} + f(x,t)$$

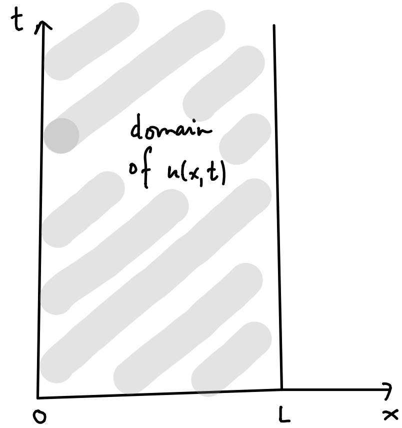
they would use $u = u_p + u_H$)

Two of the most often studied Initial boundary value problems (IBVP's)

type ①

$$\begin{cases} u_t = k u_{xx} & 0<x<L \\ & 0<t<\infty \\ u(x,0) = f(x) & \text{initial temp } 0<x<L \\ u(0,t) = 0 & t>0 \\ u(L,t) = 0 & t>0 \end{cases}$$

boundary temp. held constant (need not always be zero)



type ②

$$\begin{cases} u_t = k u_{xx} & 0<x<L \\ & 0<t<\infty \\ u(x,0) = g(x) & 0<x<L \\ u_x(0,t) = 0 & t>0 \\ u_x(L,t) = 0 & t>0 \end{cases}$$

no heat flux thru boundary (insulated end condition)

Example ① $\boxed{f(x) = \sin\left(\frac{\pi}{L}x\right)}$ or $\boxed{\sin\left(\frac{n\pi}{L}x\right)}$

product sol'n: try

$u(x,t) = v(t)\sin\omega x \qquad v(0)=1$

in the PDE

$u_t = k u_{xx}$

find $\boxed{u(x,t) = \sin\left(\frac{\pi}{L}x\right)e^{-k\left(\frac{\pi}{L}\right)^2 t}}$

solves ①. Also

$\boxed{u(x,t) = \sin\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$

general solution to ①: Use sine series for f and superpose:

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

② $g(x) = \cos\left(\frac{n\pi}{L}x\right)$

$u(x,t) = \cos\left(\frac{n\pi}{L}x\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

general solution to ②

use cosine series for g and superpose the resulting product solutions!

$$g \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}}$$

type 1 example (boundary temperature forced to be zero), with $u(x, 0) = 100,\ 0 < x < L,$ using a square wave odd extension sine series for the initial temperature:

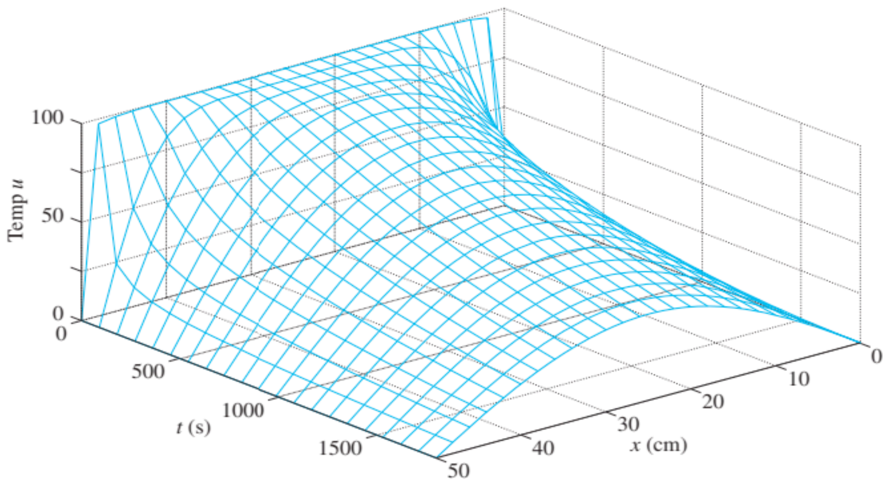


FIGURE 9.5.4. The graph of the temperature function $u(x, t)$ in Example 2.

type 2 example (zero heat flux through ends), with tent function inital data, and a tent function even extension cosine series. We'll do some computations for some examples tomorrow....

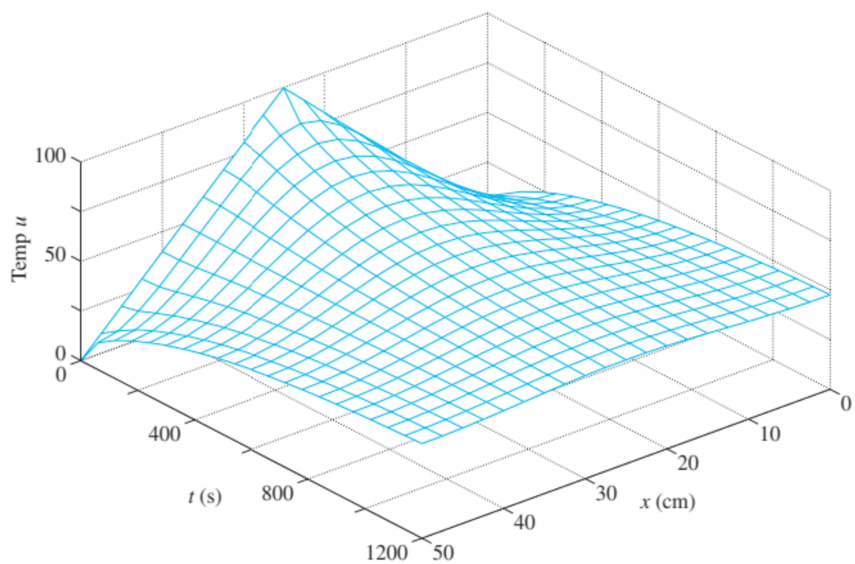


FIGURE 9.5.6. The graph of the temperature function $u(x, t)$ in Example 3.