Differentiating Fourier Series:

Theorem 3 Let f be 2π-periodic, piecewise differentiable and continuous, and with f' piecewise continuous. Let f have Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

Then f' has the Fourier series you'd expect by differentiating term by term:

$$f' \sim \sum_{n=1}^{\infty} -n\,a_n \sin(n\,t) + \sum_{n=1}^{\infty} n\,b_n \cos(n\,t)$$

proof: Let f' have Fourier series

$$f' \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \overset{nb_n}{A_n} \cos(n\,t) + \sum_{n=1}^{\infty} B_n \sin(n\,t).$$

Then

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(n\,t)\,\mathrm{d}t,\ n \in \mathbb{N}.$$

Integrate by parts with $u = \cos(n\,t)$, $dv = f'(t)dt$, $du = -n\sin(n\,t)dt$, $v = f(t)$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(n\,t)\,\mathrm{d}t = \frac{1}{\pi}\underbrace{f(t)}_{v}\underbrace{\cancel{(-n)\sin(nt)}}_{\cos nt}\Big]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(t)}_{v}\underbrace{(-n)\sin(n\,t)}_{du}\,\mathrm{d}t$$

$$= 0 + \frac{n}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t = n\,b_n$$

Similarly, $A_0 = 0$, $B_n = -n\,a_n$.

Leads to

Integrating Fourier series:

Theorem 4 Let f be 2π-periodic piecewise continuous, and let f have Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

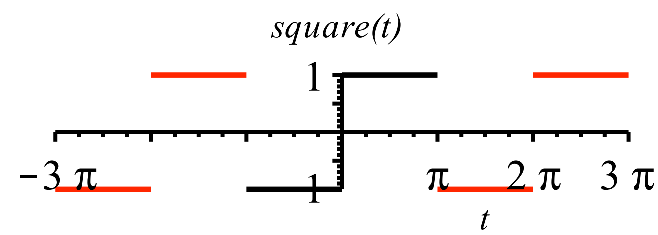
Then every antiderivative F of f is piecewise differentiable and can be found by integrating the Fourier series for f term by term:

$$F(t) = \frac{a_0}{2}t + \sum_{n=1}^{\infty} \frac{a_n}{n}\sin(n\,t) - \sum_{n=1}^{\infty} \frac{b_n}{n}\cos(n\,t) + C$$

(Note that $F(t)$ is only a periodic function if $a_0 = 0$.)

Exercise 1 On Tuesday we found the Fourier series for $sq(t)$, which is the 2π- periodic extension of

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

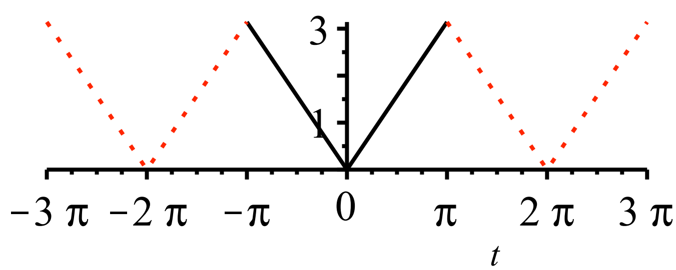


$$sq(t) = \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n} \sin(n\,t).$$

Notice that the following "tent function", *tent(t)*, is an antiderivative of $sq(t)$. $tent(t)$ is the 2π-periodic extension of $g(t) = |t|$ from the interval $[-\pi, \pi]$ to $\mathbb{R}$:

$$g(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$

did by brute force on Wed.



Find the Fourier series for $tent(t)$ by antidifferentiation. Careful with the $\frac{a_0}{2}$ term! (There's a magic identity hiding in your formula once you've got it right.)

$$tent(t) = \int sq(t)\,dt$$

$$tent(t) = \frac{4}{\pi} \sum_{n\ odd} \int \frac{1}{n} \sin nt\,dt$$

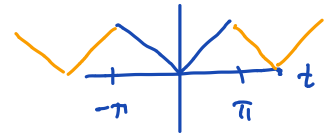
$$tent(t) = \frac{4}{\pi} \sum_{n\ odd} -\frac{\cos nt}{n^2} + \underbrace{C}_{\frac{a_0}{2} = \frac{\pi}{2}}$$

$$tent(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\ odd} \frac{\cos nt}{n^2}$$

@ $t=0$:

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n^2} \implies \frac{\pi^2}{8} = \sum_{n\ odd} \frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \dots$$

$$tent(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$



Exercise 2 For practice, find the Fourier series for $tent(t)$ by finding the Fourier coefficients directly from their definitions. You'll need to use integration by parts as well as facts about even and odd functions.

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t)$$

$$\frac{a_0}{2} := \frac{1}{2\,\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}$$

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\,t)\,\mathrm{d}t = \frac{\langle f, \cos(nt)\rangle}{\langle \cos(nt), \cos(nt)\rangle}$$

$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t = \frac{\langle f, \sin(nt)\rangle}{\langle \sin(nt), \sin(nt)\rangle}$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |t|\,dt \underset{|t| \text{ even}}{=} \frac{1}{2\pi}\, 2\int_0^{\pi} t\,dt \qquad (|t| \text{ is even.})$$

$$= \frac{1}{2\pi}\cancel{2}\left[\frac{t^2}{\cancel{2}}\right]_0^{\pi} = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|t|\cos nt}_{g(t)}\,dt = \frac{1}{\pi}\, 2\int_0^{\pi} \underbrace{t}_{u}\,\underbrace{\cos nt\,dt}_{dv}$$

$$du = dt \qquad v = \frac{\sin nt}{n}$$

$$g(-t) = |-t|\cos(-nt) = t\cos nt$$

so g is even

(product of even fcns is always even)

$$= \frac{2}{\pi}\left[\left[\frac{t\,\cancel{\sin nt}}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nt}{n}\,dt\right]$$

0

$$= \frac{2}{\pi}\left[\overset{--}{+}\frac{\cos nt}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[\cos n\pi - \cos 0\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n^2}(-2) & n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|t|\sin nt}_{g(t)}\,dt$$

$g(t)$; $g(-t) = |-t|\sin(n(-t)) = |t|(-\sin nt) = -g(t)$

(if $g(t)$ is odd, $\int_{-a}^{a} g(t)\,dt = 0$)

(product of even fcn times odd fcn is an odd fcn)

(product of odd fcn times odd fcn is even)

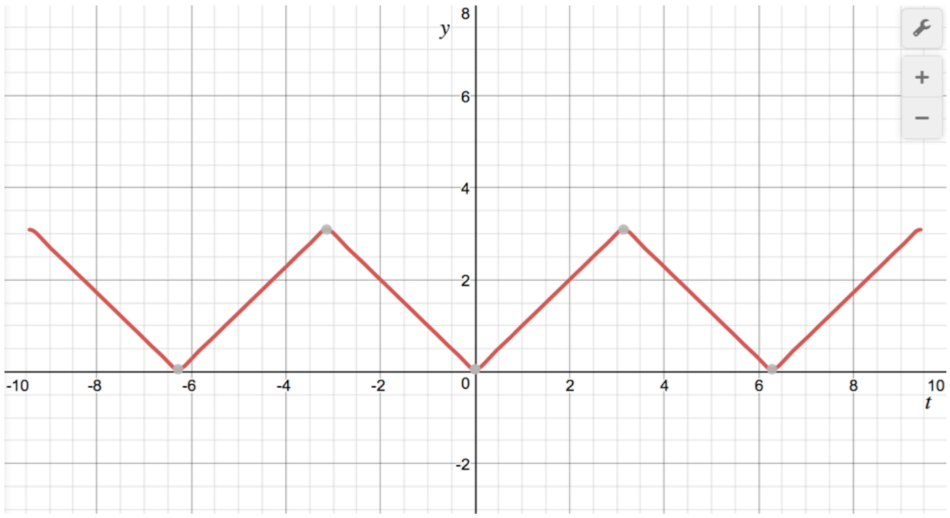
so b_n's $= 0$

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\cos nt$$

At Desmos, this typed-in command:

$$f(t)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{j=0}^{5}\frac{1}{(2\cdot j+1)^2}\cdot\cos\left((2\cdot j+1)\cdot t\right)\ \{-3\cdot\pi<t<3\cdot\pi\}$$

yielded this graph:



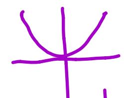
Fri April 12

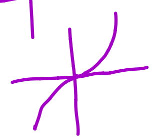
9.2-9.3 Fourier series for $2L$ - periodic functions; cosine and sine series for functions defined on the interval $[0, L]$ and extended into either even or odd $2L$ - periodic functions.

Announcements:
- typo on Wed. copy of hw13 : w13.2b evaluate series @ $t=0$, $t=\pi$ (not $t=\pi/2$)
- today :
  - differentiating/integrating Fourier series (Wed notes)
  - $2L$ - periodic fcns
  - even & odd extensions of fcns defined on $[0, L]$ to $[-L, L]$ & then $2L$-periodic

Warm-up Exercise: Check that if

a)

$f(-t) = f(t)$ f even 

$g(-t) = -g(t)$ g odd 

then

fg is odd : $f(-t)g(-t) = f(t)(-g(t)) = -f(t)g(t)$ ✓

b) f, g even $\Rightarrow$ fg even

$$f(-t)g(-t) = f(t)g(t)$$ ✓

c) f, g odd $\Rightarrow$ fg even

$$f(-t)g(-t) = (-f(t))(-g(t)) = f(t)g(t)$$ ✓

Fourier series for $2L$ - periodic functions:

So far we've only talked about Fourier series for 2π-periodic functions. In applications we want to be able to vary the period, and consider $2L$ - periodic functions instead, where L can be specified in the application. There's no problem in doing so:

<u>Theorem</u> Let $V = \{f : \mathbb{R} \to \mathbb{R}\ s.t.\ f$ is piecewise continuous and $2L$ — periodic $\}$.

2L - periodic

Define the Fourier series for f by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}\,t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}\,t\right)$$

$$\cos\left(\frac{n\pi}{L}(t+2L)\right) = \cos\left(\frac{n\pi}{L}t + 2n\pi\right) = \cos\left(\frac{n\pi}{L}t\right) \checkmark$$

where the Fourier coefficients of f are defined analogously as for the 2π - periodic case. Note that the Fourier coefficients can again be expressed as projection weights with respect to an (adapted) inner product

$$\langle f, g \rangle := \int_{-L}^{L} f(t) g(t)\, dt.$$

$$\frac{a_0}{2} := \frac{1}{2L} \int_{-L}^{L} f(t)\, dt = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \text{the average value of } f.$$

$$a_n := \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{n\pi}{L}\,t\right) dt = \frac{\left\langle f, \cos\left(\frac{n\pi}{L}t\right)\right\rangle}{\left\langle \cos\left(\frac{n\pi}{L}t\right), \cos\left(\frac{n\pi}{L}t\right)\right\rangle}$$

$$b_n := \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{n\pi}{L}\,t\right) dt = \frac{\left\langle f, \sin\left(\frac{n\pi}{L}t\right)\right\rangle}{\left\langle \sin\left(\frac{n\pi}{L}t\right), \sin\left(\frac{n\pi}{L}t\right)\right\rangle}$$

So the truncated Fourier series is the projection of f onto the $2N+1$ dimensional subspace

$$V_N := span\left\{1, \cos\left(\frac{\pi}{L}\,t\right), \cos\left(\frac{2\pi}{L}t\right), \ldots, \cos\left(\frac{N\pi}{L}t\right), \sin\left(\frac{\pi}{L}t\right), \sin\left(\frac{2\pi}{L}t\right), \ldots \sin\left(\frac{N\pi}{L}t\right)\right\}$$

$$proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi}{L}\,t\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{n\pi}{L}\,t\right).$$

The same convergence theorems, and integration/differentiation theorems hold as for the 2π - periodic case.

One reason the same theorems hold for the $2L-$ periodic functions and their Fourier series, as for the $2\pi-$periodic ones, is because it's possible to change the periods of functions by scaling the input variables, and relate the corresponding facts that way:

Let f, g be $2L-$periodic, with the inner product

$$\langle f, g\rangle := \int_{-L}^{L} f(t)g(t)\,\mathrm{d}t.$$

Change variables, letting

$$t = \frac{L}{\pi}\tau, \qquad dt = \frac{L}{\pi}\,d\tau$$

$t = \frac{L}{\pi}\tau$

$-\pi \quad \tau \quad \pi \qquad -L \quad t \quad L$

$\tau = \frac{\pi}{L}t$

Then $-L \le t \le L$ corresponds to $-\pi \le \tau \le \pi$. In terms of the inner products,

$$\int_{-L}^{L} f(t)g(t)\,\mathrm{d}t = \frac{L}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}\tau\right)g\left(\frac{L}{\pi}\tau\right)\mathrm{d}\tau$$

$t = \frac{L}{\pi}\tau$

$dt = \frac{L}{\pi}\,d\tau$

$$\frac{1}{L}\int_{-L}^{L} f(t)g(t)\,\mathrm{d}t = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}\tau\right)g\left(\frac{L}{\pi}\tau\right)\mathrm{d}\tau.$$

In particular,

1) The $2L-$ periodic functions

$$\left\{1, \cos\left(\frac{\pi}{L}t\right), \cos\left(\frac{2\pi}{L}t\right), \ldots, \sin\left(\frac{\pi}{L}t\right), \sin\left(\frac{2\pi}{L}t\right), \ldots\right\}$$

correspond to the $2-\pi$ periodic functions

$$\{1, \cos(\tau), \cos(2\tau), \ldots, \sin(\tau), \sin(2\tau), \ldots\}$$

and the first collection is orthogonal with respect to the $2L-$ periodic function inner product because the second collection is orthogonal with respect to the $2\pi-$periodic function inner product.

Wed.

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi}{L}t\right)\cos\left(\frac{m\pi}{L}t\right)dt = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\left(\frac{n\pi}{L}\cdot\frac{L}{\pi}\tau\right)\cos(m\tau)\,d\tau = 0$$

$m \neq n$

2) If $f(t)$ is $2L-$periodic, then its Fourier coefficients are the same as those for the $2\pi-$periodic function $f\left(\frac{L}{\pi}\tau\right)$; If $g(\tau)$ is $2\pi-$periodic, then its Fourier coefficients are the same as those for the $2L-$periodic function $g\left(\frac{\pi}{L}t\right)$.

a_n for $f(t)$: $\frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt$

$t = \frac{L}{\pi}\tau$

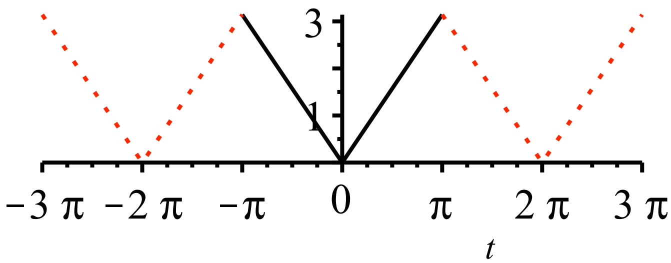
$= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}\tau\right)\cos(n\tau)\,d\tau$

$= a_n$ for 2π-periodic fcn

$f\left(\frac{L}{\pi}\tau\right)$

Exercise 1 Use the Fourier series for 2π- tent function to find the Fourier series for a tent function with period 2. Careful! (But if you do it right you save a lot of time over recomputing all of the Fourier coefficients using the formulas for $2L$-periodic functions!)

$$tent(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$



$-\pi \quad \tau \quad \pi$

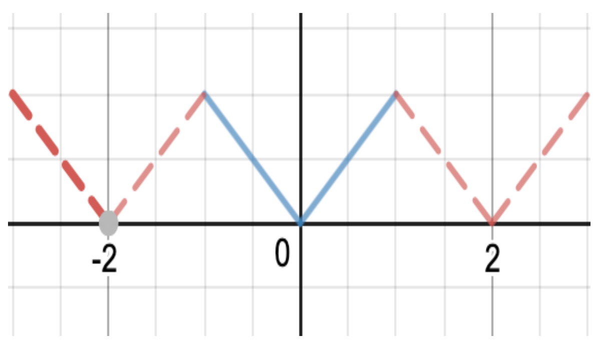
$$tent(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n^2} \cos(n\,t)$$

$\tau = \pi t$

$$f(t) = \begin{cases} -t & -1 < t < 0 \\ t & 0 < t < 1 \end{cases} = |t|$$

$-1 \quad t \quad 1$

on $[-1,1]$



$f(t) \overset{?}{=} tent(\pi t)$

$f(t) \overset{?}{=} |\pi t| = |\pi||t|$

not quite

So

$f(t) = \frac{1}{\pi}\, tent(\pi t)$

$= \frac{1}{\pi} |\pi t| = |t|$ ✓

So $f(t) = \frac{1}{\pi}\, tent(\pi t) = \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{4}{\pi}\sum_{n\ odd} \frac{1}{n^2}\cos(n\pi t)\right)$

from above

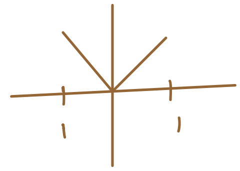
$$\boxed{f(t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n\ odd} \frac{1}{n^2}\cos(n\pi t)}$$

$\cos\left(\frac{n\pi}{L} t\right)$

$L = 1$

brute force. $f(t) = |t| \qquad -1 \le t \le 1$

$$\frac{a_0}{2} = \frac{1}{2}\int_{-1}^{1} |t|\,dt = \frac{1}{2}$$



$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right)dt \qquad = \frac{2}{L}\int_0^L \underbrace{\underbrace{f(t)}_{\text{even}}\underbrace{\cos\left(\frac{n\pi}{L}t\right)}_{\text{even}}}_{\text{even}}\,dt$$

f even, cos even, $L = 1$

$$a_n = 2\int_0^1 \underbrace{t}_{u}\,\underbrace{\cos(n\pi t)\,dt}_{dv}$$

$$du = dt \qquad v = \frac{\sin(n\pi t)}{n\pi}$$

$$a_n = 2\left[\underbrace{\left[t\,\frac{\sin(n\pi t)}{n\pi}\right]_0^1}_{0} - \int_0^1 \frac{\sin(n\pi t)}{n\pi}\,dt\right]$$

$$= 2\left[\frac{\cos(n\pi t)}{(n\pi)^2}\right]_0^1$$

$$= \frac{2}{(n\pi)^2}\left(\cos n\pi - \cos 0\right) = \begin{cases} 0 & n \text{ even} \\ -\dfrac{4}{(n\pi)^2} & n \text{ odd} \end{cases}$$

Same! as previous page

b_n's turn out to be zero because $|t|$ is even

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(t)}_{\text{even}}\,\underbrace{\sin\left(\frac{n\pi}{L}t\right)}_{\text{odd}}}_{\text{odd}}\,dt = 0$$