So what's going on?

<u>Theorem</u> Let $V = \{ f : \mathbb{R} \to \mathbb{R} \ s.t. \ f$ is piecewise continuous and $2\pi$ - periodic $\}$. Define

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(t) g(t) \, \mathrm{d}t.$$

1) Then $V, \langle \ , \ \rangle$ is an inner product space.
2) Let $V_N := span\{1, \cos(t), \cos(2t), ..., \cos(Nt), \sin(t), \sin(2t), ... \sin(Nt)\}$. Then the $2N + 1$ functions listed in this collection are an orthogonal basis for the $(2N + 1)$ dimensional subspace $V_N$. In particular, for any $f \in V$ the nearest function in $V_N$ to $f$ is given by the projection formula

$\mathbb{R}^n$ $\quad proj_V \vec{x} = \frac{\vec{x} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \cdots + \frac{\vec{x} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k} \vec{v}_k$ $\quad$ if $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is orthogonal basis for $V$

$$proj_{V_N} f = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} 1 + \sum_{n=1}^{N} \frac{\langle f, \cos(nt) \rangle}{\langle \cos(nt), \cos(nt) \rangle} \cos(nt) + \sum_{n=1}^{N} \frac{\langle f, \sin(nt) \rangle}{\langle \sin(nt), \sin(nt) \rangle} \sin(nt)$$

Fourier series $\quad f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$

and this works out to be precisely the truncated Fourier series

$$proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

where $a_0, a_n, b_n$ are the Fourier coefficients defined earlier:

$$\frac{a_0}{2} := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, \mathrm{d}t = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_{-\pi}^{\pi} f(t) \cdot 1 \, dt}{\int_{-\pi}^{\pi} 1 \, dt} \qquad \int_{-\pi}^{\pi} 1 \, dt = 2\pi \quad \checkmark$$

$$a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, \mathrm{d}t = \frac{\langle f, \cos(nt) \rangle}{\langle \cos(nt), \cos(nt) \rangle} = \frac{\int_{-\pi}^{\pi} f(t) \cos nt \, dt}{\int_{-\pi}^{\pi} \cos^2 nt \, dt}$$

$$b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \, \mathrm{d}t = \frac{\langle f, \sin(nt) \rangle}{\langle \sin(nt), \sin(nt) \rangle}$$

$\int_{-\pi}^{\pi} \cos^2 nt \, dt = (2\pi)(\frac{1}{2}) = \pi$

$$= \frac{\int_{-\pi}^{\pi} f(t) \sin nt \, dt}{\underbrace{\int_{-\pi}^{\pi} \sin^2 nt \, dt}_{\frac{1}{2}(2\pi) = \pi}}$$

$\cos^2 nt + \sin^2 nt = 1$
average value of $\cos^2 nt + \sin^2 nt = 1$

average value of $\cos^2 nt = 1/2$
(on any interval whose length is a multiple of its period)

Exercise 2) Partially check that $\{1,\ \cos(t), \cos(2\,t), ..., \cos(N\,t), \sin(t), \sin(2\,t), ...\ \sin(N\,t)\}$ is orthogonal for our inner product, and also check why the Fourier coefficients match up to the inner product expressions.

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(t)g(t)\ \mathrm{d}t \overset{?}{=} 0$$ when $f \neq g$ are chosen from the list above.

Hint:

$$\cos((m+k)t) = \cos(m\,t)\cos(k\,t) - \sin(m\,t)\sin(k\,t)$$
$$\sin((m+k)t) = \sin(m\,t)\cos(k\,t) + \cos(m\,t)\sin(k\,t)$$

so

- $\cos(m\,t)\cos(k\,t) = \frac{1}{2}(\cos((m+k)\,t) + \cos((m-k)t))$ (even if $m = k$)

$$\sin(m\,t)\sin(k\,t) = \frac{1}{2}(\cos((m-k)\,t) - \cos((m+k)t))\quad \text{(even if } m = k)$$

$$\cos(m\,t)\sin(k\,t) = \frac{1}{2}(\sin((m+k)\,t) + \sin((-m+k)t))\quad \text{(even if } m = k)$$

$\langle \cos mt, \cos kt\rangle = 0 \qquad m \neq k:$

$$\int_{-\pi}^{\pi} \cos(mt)\cos(kt)\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+k)t + \cos(m-k)t\ dt$$

$$= \frac{1}{2}\left[\frac{\sin(m+k)t}{m+k} + \frac{\sin(m-k)t}{m-k}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(0 - 0 + (0-0)\right)$$

$\sin k\pi = 0 \quad k \in \mathbb{Z}.$

Convergence Theorems (These require some careful mathematical analysis to prove - they are often discussed in Math 5210, for example.)

Theorem 1 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be $2\pi$ - periodic and piecewise continuous. Let

$$f_N = proj_{V_N} f = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

be the Fourier series truncated at $N$. Then

delete

$$\lim_{n \to \infty} \|f - f_N\| = \lim_{n \to \infty} \left[ \cancel{\frac{1}{\pi}} \int_{-\pi}^{\pi} \left(f(t) - f_N(t)\right)^2 \, dt \right]^{\frac{1}{2}} = 0.$$

In other words, the distance between $f_N$ and $f$ converges to zero, where we are using the distance function that we get from the inner product,

$$dist(f, g) = \|f - g\| = \sqrt{\langle f - g, f - g \rangle} = \left( \int_{-\pi}^{\pi} (f(t) - g(t))^2 \, dt \right)^{\frac{1}{2}}.$$

Theorem 2 If $f$ is as in Theorem 1, and is (also) piecewise differentiable with at most jump discontinuities, then

(i) for any $t_0$ such that $f$ is differentiable at $t_0$

$$\lim_{N \to \infty} f_N(t_0) = f(t_0) \quad \text{(pointwise convergence).}$$

(ii) for any $t_0$ where $f$ is not differentiable (but is either continuous or has a jump discontinuity), then

$$\lim_{N \to \infty} f_N(t_0) = \frac{1}{2} \left( f_-(t_0) + f_+(t_0) \right)$$

where

$$f_-(t_0) = \lim_{t \to t_0^-} f(t), \quad f_+(t_0) = \lim_{t \to t_0^+} f(t)$$

Example: The truncated Fourier series for the square wave, i.e. the $sq_N(t)$, converge to $sq(t)$ for all $t$ which are not multiples of $\pi$. At integer multiples of $\pi$ the partial sums are all zero, and so is the limit. Zero is the average of the left and right hand limits of $sq(t)$ at these jump discontinuities.

Math 3150, 3140
5210
5440
5710

Wed April 10 • Continue Fourier series

9.1-9.3 Differentiating and integrating Fourier series.

Announcements:

• pick up new HW assignment (§9.1–9.3)
There will be one more! (§9.4-9.6)

• Quiz today: Compute $e^{tA}$ for $A_{2\times 2}$ diagonalizable (real eigendata :))

• Later classes which discuss Fourier series & applications:

3140/3150 "Intro to PDE's"
5210 "Intro to real analysis"
5440 "Intro to PDE's"
5710 "Intro to applied math"

Warm-up Exercise: nope

Differentiating Fourier Series:

Theorem 3 Let $f$ be $2\pi$-periodic, piecewise differentiable and continuous, and with $f'$ piecewise continuous. Let $f$ have Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

Then $f'$ has the Fourier series you'd expect by differentiating term by term:

$$f' \sim \sum_{n=1}^{\infty} -n\,a_n \sin(n\,t) + \sum_{n=1}^{\infty} n\,b_n \cos(n\,t)$$

proof: Let $f'$ have Fourier series

$$f' \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\,t) + \sum_{n=1}^{\infty} B_n \sin(n\,t).$$

Then

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(n\,t)\,\mathrm{d}t,\ n \in \mathbb{N}.$$

Integrate by parts with $u = \cos(n\,t)$, $dv = f'(t)dt$, $du = -n\sin(n\,t)dt$, $v = f(t)$:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos(n\,t)\,\mathrm{d}t = \frac{1}{\pi} f(t)(-n)\sin(nt)\,]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(-n)\sin(n\,t)\,\mathrm{d}t$$

$$= 0 + \frac{n}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t = n\,b_n.$$

Similarly, $A_0 = 0$, $B_n = -n\,a_n$.

Leads to

Integrating Fourier series:

Theorem 4 Let $f$ be $2\pi$-periodic piecewise continuous, and let $f$ have Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t).$$

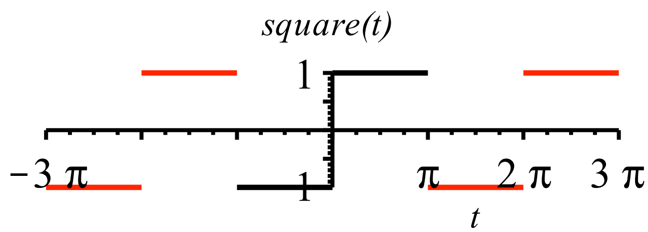
Then every antiderivative $F$ of $f$ is piecewise differentiable and can be found by integrating the Fourier series for $f$ term by term:

$$F(t) = \frac{a_0}{2}t + \sum_{n=1}^{\infty} \frac{a_n}{n}\sin(n\,t) - \sum_{n=1}^{\infty} \frac{b_n}{n}\cos(n\,t) + C$$

(Note that $F(t)$ is only a periodic function if $a_0 = 0$.)

Exercise 1  On Tuesday we found the Fourier series for $sq(t)$, which is the $2\pi-$ periodic extension of

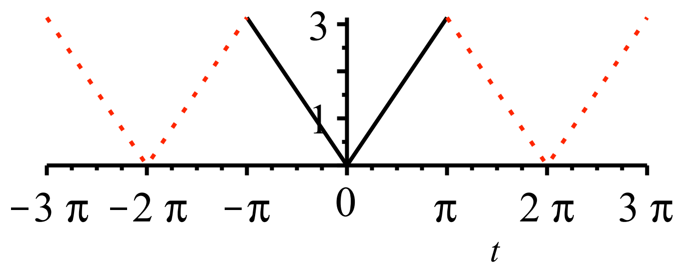
$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$




$$sq(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(n\,t).$$

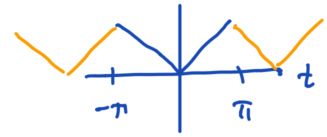
Notice that the following "tent function", *tent(t)*, is an antiderivative of $sq(t)$.  $tent(t)$ is the $2\pi-$periodic extension of $g(t) = |t|$ from the interval $[-\pi, \pi]$ to $\mathbb{R}$:

$$g(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$




Find the Fourier series for $tent(t)$ by antidifferentiation.  Careful with the $\frac{a_0}{2}$ term!  (There's a magic identity hiding in your formula once you've got it right.)

$$\text{tent}(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$




Exercise 2 For practice, find the Fourier series for $tent(t)$ by finding the Fourier coefficients directly from their definitions. You'll need to use integration by parts as well as facts about even and odd functions.

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t)$$

$$\frac{a_0}{2} := \frac{1}{2\,\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}$$

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\,t)\,\mathrm{d}t = \frac{\langle f, \cos(nt)\rangle}{\langle \cos(nt), \cos(nt)\rangle}$$

$$b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\,t)\,\mathrm{d}t = \frac{\langle f, \sin(nt)\rangle}{\langle \sin(nt), \sin(nt)\rangle}$$

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |t|\,dt \underset{|t| \text{ even}}{=} \frac{1}{2\pi}\, 2\int_0^{\pi} t\,dt \qquad (|t| \text{ is even.})$$

$$= \frac{1}{2\pi}\,\cancel{2}\left[\frac{t^2}{\cancel{2}}\right]_0^{\pi} = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|t| \cos nt}_{g(t)}\,dt = \frac{1}{\pi}\, 2\int_0^{\pi} \underbrace{t}_{u}\, \underbrace{\cos nt\,dt}_{dv}$$

$g(-t) = |-t|\cos(-nt)$
$= t \cos nt$

So $g$ is even

(product of even fcns is always even)

$du = dt \quad v = \dfrac{\sin nt}{n}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|t| \sin nt}_{g(t)}\,dt$$

$g(t)$; $g(-t) = |-t| \sin(n(-t))$
$= |t|(-\sin nt)$
$= -g(t)$

$$= \frac{2}{\pi}\left[ \cancel{\left[\frac{t \sin nt}{n}\right]_0^{\pi}}^{\,0} - \int_0^{\pi} \frac{\sin nt}{n}\,dt \right]$$

(if $g(t)$ is odd, $\int_{-a}^{a} g(t)\,dt = 0$)

$$= \frac{2}{\pi}\left[ + \frac{\cos nt}{n^2}\right]_0^{\pi}$$

(product of even fcn times odd fcn is an odd fcn)

So $b_n$'s $= 0$

(product of odd fcn times odd fcn is even)

$$= \frac{2}{\pi n^2}\left[\cos n\pi - \cos 0\right]$$

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\cos nt$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi n^2}(-2) & n \text{ odd} \end{cases}$$

At Desmos, this typed-in command:

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{5} \frac{1}{(2 \cdot j + 1)^2} \cdot \cos\left((2 \cdot j + 1) \cdot t\right) \ \{-3 \cdot \pi < t < 3 \cdot \pi\}$$

yielded this graph: