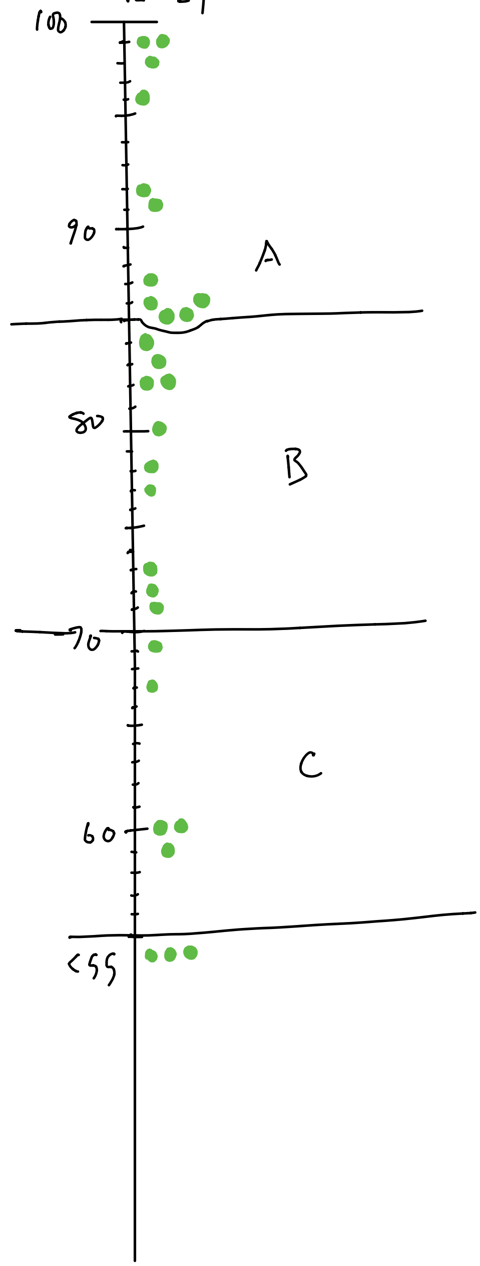
Exam 2 scores
2280-2
Spring '19
n=29



Math 2280-002
Week 12, April 1-4, 5.4, 5.6-5.7
Continuing discussion of unforced and forced mass-spring systems; matrix exponentials and applications.

Mon April 1

5.4 mass-spring systems, and forced oscillations

Announcements:
- I'm happy to set up an appointment with anyone who wants to discuss exam or class.
- Our plan:
  - rest of week 12 after today: 5.6-5.7 Matrix exponentials & applications
  - weeks 13-15 (2 ½ weeks) Fourier series & partial differential eqtns ("PDE"s)

(Not planning to do Laplace Transform)

Warm-up Exercise: Suppose $\vec{v} \in Nul(A) = \underline{E_{\lambda=0}}$. $A\vec{v} = 0\vec{v} = \vec{0}$

Show that

$$\vec{x}(t) = (c_1 + c_2 t)\vec{v}$$

solves the second order mass-spring system

$$\boxed{\vec{x}''(t) = A\vec{x}.}$$

$\vec{x}(t) = (c_1 + c_2 t)\vec{v}$

LHS: $\vec{x}''(t) = (c_1 + c_2 t)''\vec{v}$ (product rule... constant mult rule, here $\vec{v}$ is a const. vec)

$= 0\vec{v} = \vec{0}$

RHS $A\vec{x} = A(c_1 + c_2 t)\vec{v}$

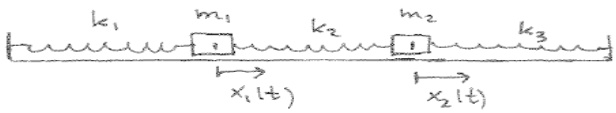
scalar fcn

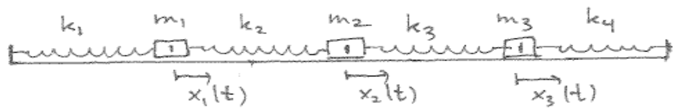
$= (c_1 + c_2 t)\underbrace{A\vec{v}}_{\vec{0}}$ $\vec{v} \in Nul A.$

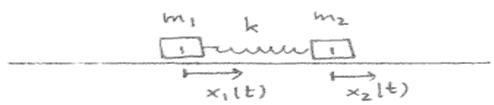
$= \vec{0}$

LHS = RHS so $\vec{x}(t)$ are solns

Before the exam we were studying unforced mass-spring systems ...







Newton's law leads to a second order system of differential equations for the vector of displacements of the masses from their equilibrium locations,

$$M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}.$$

If there are n masses the solution space is $2\,n$- dimensional, because for each mass you can specify initial displacement and velocity in the IVP.

<u>Solution space algorithm:</u> Consider the homogeneous system of linear differential equations,

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

<u>If</u> $A_{n \times n}$ is a diagonalizable matrix and <u>if all of its eigenvalues are non-positive</u> then for each <u>eigenpair</u> $\left(\lambda_j, \underline{\boldsymbol{v}}_j\right)$ with $\lambda_j < 0$ there are two linearly independent sinusoidal solutions to $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$ given by

$$\boldsymbol{x}_j(t) = \cos\left(\omega_j\,t\right)\underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = \sin\left(\omega_j\,t\right)\underline{\boldsymbol{v}}_j$$

same plug-in method as in warmup.

with

$\omega_j^2 = -\lambda_j$

$$\omega_j = \sqrt{-\lambda_j}\ .$$

LHS: $\frac{d^2}{dt^2}\left(\cos\omega_j t\right)\vec{v}_j = -\omega_j^2 \cos\omega_j t\,\vec{v}_j$

And for an eigenpair $\left(\lambda_j, \underline{\boldsymbol{v}}_j\right)$ with $\lambda_j = 0$ there are two indpendent solutions given by constant and linear functions

RHS: $A\left(\cos\omega_j t\right)\vec{v}_j$

$$\boldsymbol{x}_j(t) = \underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = t\,\underline{\boldsymbol{v}}_j$$

$= \cos\omega_j t\,\underbrace{A\vec{v}_j}_{\lambda_j \vec{v}_j}$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$, i.e. a basis for the solution space.

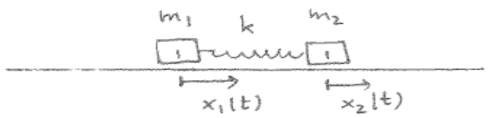
$= \cos\omega_j t\left(-\omega_j^2\,\vec{v}_j\right)$

LHS = RHS ✓

<u>Remark</u>: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall, all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution solutions $\left(c_1 + c_2\,t\right)\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

We had analyzed a two-mass, three spring configuration in which the masses and springs were all the same, and conducted an experiment which tested the model. We were in the middle of this exercise, although last Wednesday we set $m_1 = m_2$ for simplicity ...

Exercise 1) Consider a train with two cars connected by a spring:



$$m_1 x_1'' = k(x_2 - x_1)$$
$$m_2 x_2'' = -k(x_2 - x_1)$$

1a) Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1'' = \frac{k}{m_1}(x_2 - x_1)$$
$$x_2'' = -\frac{k}{m_2}(x_2 - x_1)$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1b). Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ remember that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in your homework problem about CO_2 vibrations.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_3 \cos\omega_2 t + c_4 \sin\omega_2 t)\begin{bmatrix} -\frac{m_2}{m_1} \\ 1 \end{bmatrix}$$

translation "mode"

out of phase mode

$m_2 = m_1 \Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Input:

eigenvalues	$\begin{pmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{pmatrix}$

Results:

$\lambda_1 = 0$

$\lambda_2 = \frac{k(-m_1 - m_2)}{m_1 m_2} = -k\left(\frac{m_1 + m_2}{m_1 m_2}\right)$

$\omega_2 = \sqrt{-\lambda_2} = \sqrt{\frac{m_1 + m_2}{m_1 m_2}k}$

Corresponding eigenvectors:

$v_1 = (1, 1)$

$v_2 = \left(-\frac{m_2}{m_1}, 1\right)$

Forced oscillations (still undamped):

$$M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{F}}(t)$$
$$\Rightarrow \underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + M^{-1}\underline{\boldsymbol{F}}(t)\ .$$

If the forcing is sinusoidal,

$$M\,\underline{\boldsymbol{x}}''(t) = K\,\underline{\boldsymbol{x}} + \cos(\omega\,t)\underline{\boldsymbol{G}}_0$$
$$\Rightarrow\ \underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + \cos(\omega\,t)\underline{\boldsymbol{F}}_0$$

with $\underline{\boldsymbol{F}}_0 = M^{-1}\underline{\boldsymbol{G}}_0$.

From vector space theory we know that the general solution to this inhomogeneous linear problem is of the form

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_P(t) + \underline{\boldsymbol{x}}_H(t)\ ,$$

Forced oscillation particular solution algorithm:

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + \cos(\omega\,t)\underline{\boldsymbol{F}}_0$$

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\underline{\boldsymbol{x}}_P(t) = \cos(\omega\,t)\ \underline{\boldsymbol{d}}$$

where the constant vector $\underline{\boldsymbol{d}}$ is what we need to find. (It's value will depend on the angular frequency ω of the forcing function.)

Exercise 2) Substitute the guess $\underline{\boldsymbol{x}}_P(t) = \cos(\omega\,t)\ \underline{\boldsymbol{d}}$ into the DE system

$$\boxed{\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} + \cos(\omega\,t)\underline{\boldsymbol{F}}_0}$$

to find a matrix algebra formula for $\underline{\boldsymbol{d}} = \underline{\boldsymbol{d}}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

$\vec{x}_p = \cos\omega t\ \vec{d}$

LHS: $-\omega^2\cos\omega t\ \vec{d}$

RHS: $A\cos\omega t\ \vec{d} + \cos\omega t\ \vec{F}_0$

$\cos\omega t\ (A\vec{d} + \vec{F}_0)$

LHS=RHS iff $-\omega^2\vec{d} = A\vec{d} + \vec{F}_0$

$-\vec{F}_0 = A\vec{d} + \omega^2\vec{d} = (A+\omega^2 I)\vec{d}$

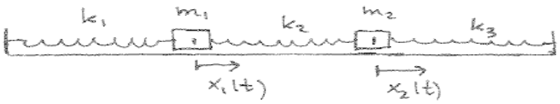
mult by $(A+\omega^2 I)^{-1}$ on left

Solution: $\underline{\boldsymbol{d}}(\omega) = -\left(A + \omega^2 I\right)^{-1}\underline{\boldsymbol{F}}_0$. Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue, i.e. ω is not one of the natural frequencies.

$(A+\omega^2 I)$ is invertible iff $|A+\omega^2 I| \neq 0$

$(A-\lambda I)$ is invertible iff $|A-\lambda I| \neq 0$

i.e. $-\omega^2$ is not one of the eigenvalues



$k_1, k_2, k_3 = k, \quad m_1 = m_2 = m$

Last week experimental configuration model:

$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

$$m x_1'' = -2k x_1 + k x_2$$
$$m x_2'' = +k x_1 - 2k x_2$$

Exercise 3) Continuing with the configuration shown above, but now for a forced oscillation problem, let $k = m$ (one can do this by changing units of time to make the discussion completely general and force the second mass sinusoidally:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix} \longleftarrow \text{forcing } 2^{\text{nd}} \text{ mass only.}$$

We know from previous work that the natural frequencies are $\omega_1 = \sqrt{\frac{k}{m}} = 1$, $\omega_2 = \sqrt{\frac{3k}{m}} = \sqrt{3}$ and that

$$\underline{\boldsymbol{x}}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos\left(\sqrt{3}\, t - \alpha_2\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\underline{\boldsymbol{x}}_P(t)$, using the undetermined coefficients formula

$$\underline{\boldsymbol{d}}(\omega) = -\left(A + \omega^2 I\right)^{-1} \underline{\boldsymbol{F}}_0$$

Notice that this steady periodic solution blows up as $\omega \to 1$ or $\omega \to \sqrt{3}$. (If we don't have time to work this by hand, we may skip directly to the technology check on the next page. But since we have quick formulas for inverses of 2 by 2 matrices, this is definitely a computation we could do by hand.)

$$\vec{x}_P = \cos\omega t \ \vec{d}$$

$$\vec{d}(\omega) = -\begin{bmatrix} -2+\omega^2 & 1 \\ 1 & -2+\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{(\omega^2-2)^2 - 1} \begin{bmatrix} -2+\omega^2 & -1 \\ -1 & -2+\omega^2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$= \frac{1}{(\omega^2-3)(\omega^2-1)} \begin{bmatrix} 3 \\ -3\omega^2+6 \end{bmatrix}$$

note $\omega=\sqrt{3}$
$\omega=1$
no go.
because no x_{sp}
in this case

Solution: As long as $\omega \neq 1, \sqrt{3}$, the general solution $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t) \begin{bmatrix} \dfrac{3}{(\omega^2-1)(\omega^2-3)} \\ \dfrac{6-3\omega^2}{(\omega^2-1)(\omega^2-3)} \end{bmatrix}$$

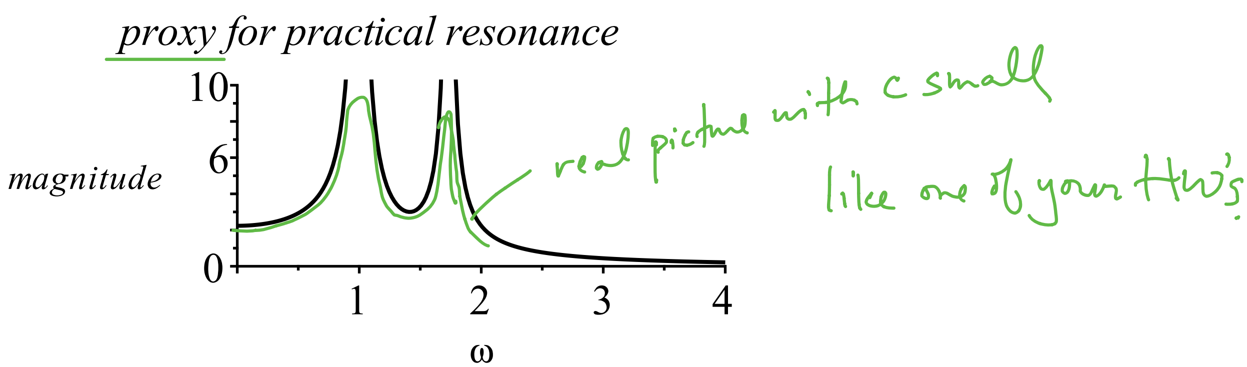
$$+ \; C_1 \cos(t-\alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\,t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

<u>Interpretation</u> as far as inferred <u>practical resonance</u> for <u>slightly damped</u> problems: If there was even a small amount of damping, the homogeneous solution would actually be transient (it would be exponentially decaying and oscillating - underdamped). There would still be a sinusoidal particular solution, which would have a formula close to our particular solution, the first term above, as long as $\omega \neq 1, \sqrt{3}$. (There would also be a relatively smaller $\sin(\omega t)\tilde{\underline{\boldsymbol{d}}}$ term as well.) So we can infer the practical resonance behavior for different ω values with slight damping, by looking at the size of the $\underline{\boldsymbol{c}}(\omega)$ term for the undamped problem....see next page for visualizations.

```
> restart :
> with(LinearAlgebra) :
> A := Matrix(2, 2, [-2, 1, 1, -2]) :
> F0 := Vector([0, 3]) :
> Iden := IdentityMatrix(2) :
> d := ω → (A + ω^2·Iden)^(-1).(-F0) :  # the formula we worked out by hand
> d(ω);
```

$$\begin{bmatrix} \dfrac{3}{\omega^4 - 4\,\omega^2 + 3} \\ -\dfrac{3\,(\omega^2 - 2)}{\omega^4 - 4\,\omega^2 + 3} \end{bmatrix} \qquad \textbf{(1)}$$

```
> with(plots) :
  with(LinearAlgebra) :
> plot(Norm(d(ω), 2), ω = 0 ..4, magnitude = 0 ..10, color = black, title = `proxy for practical resonance`);
      # Norm(c(ω),2) is the magnitude of the c(ω) vector
```




```
>
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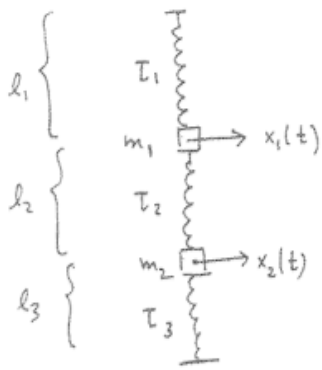
There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings...this is like one of your homework problems from a few weeks ago...

• Transverse oscillations! (i.e. directions $\perp$ to the mass-spring configuration)



T_1, T_2, T_3 are the tensions (pulling forces) of the stretched springs

By linearization, a good model would be

$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1+K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = K_2(x_1 - x_2) - K_3 x_2 = K_2 x_1 \quad -(K_2+K_3)x_2$$

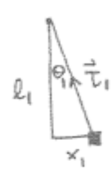
where $K_1, K_2\, K_3$ are positive constants as before

→ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

• upshot : transverse oscillations satisfy analogous systems of 2^{nd} order linear DE's ; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & ~~fu~~ fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:

[force picture, e.g.



horiz force from top spring on mass 1

$$= -T_1 \sin\theta_1 = -T_1 \frac{x_1}{\sqrt{\ell_1^2 + x_1^2}} \approx -T_1 \frac{x_1}{\ell_1} = -\frac{T_1}{\ell_1} x_1$$

So $K_1 = \frac{T_1}{\ell_1}$

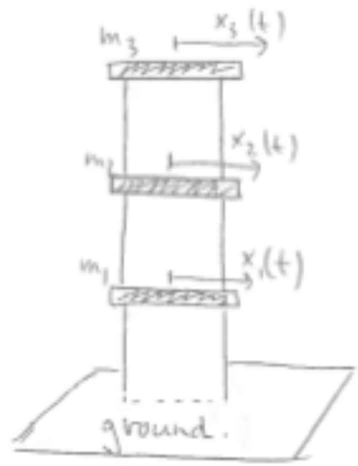
similarly, $K_2 = \frac{T_2}{\ell_2}$, $K_3 = \frac{T_3}{\ell_3}$

for our physics demo springs, equilibrium length ≈ 0, very Hookesian so $T \cong k\ell$; $\frac{T}{\ell} \approx k$, so actually almost recover same fundamental modes !!]

- An interesting shake-table demonstration:

http://www.youtube.com/watch?v=M_x2jOKAhZM

Below is a discussion of how to model the unforced "three-story" building shown shaking in the video above, from which we can see which modes will be excited. There is also a "two-story" building model in the video, and its matrix and eigendata follow. Here's a schematic of the three-story building:



For the unforced (homogeneous) problem, the accelerations of the three massive floors (the top one is the roof) above ground and of mass m, are given by

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Note the -1 value in the last diagonal entry of the matrix. This is because $x_3(t)$ is measuring displacements for the top floor (roof), which has nothing above it. The "k" is just the linearization proportionality factor, and depends on the tension in the walls, and the height between floors, etc, as discussed on the previous page.

Exercise 4 Here is eigendata for the unscaled matrix $\left(\frac{k}{m}=1\right)$. For the scaled matrix you'd have the same eigenvectors, but the eigenvalues would all be multiplied by the scaling factor $\frac{k}{m}$ and the natural frequencies would all be scaled by $\sqrt{\frac{k}{m}}$ but the eigenvectors describing the modes would stay the same. Use this information describe the fundamental modes, and the order in which they will appear.

Input:

eigenvalues $\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Results:

$\lambda_1 \approx -3.24698$ $\omega_1 = 1.802$

$\lambda_2 \approx -1.55496$ $\omega_2 = 1.247$

$\lambda_3 \approx -0.198062$ $\omega_3 = .445$

Corresponding eigenvectors:

$v_1 \approx (1.80194, -2.24698, 1)$

$v_2 \approx (-1.24698, -0.554958, 1)$

$v_3 \approx (0.445042, 0.801938, 1)$

Input:

eigenvalues $\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$

Results:

$\lambda_1 \approx -2.61803$ $\omega_1 = 1.618$

$\lambda_2 \approx -0.381966$ $\omega_2 = .618$

Corresponding eigenvectors:

$v_1 \approx (-1.61803, 1)$

$v_2 \approx (0.618034, 1)$