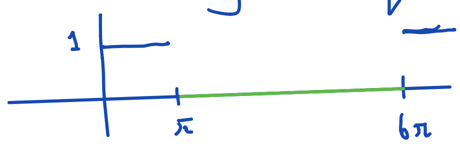
Tues April 23
9.6 wave equation solutions odds and ends. Course review

Announcements: • Please hand HW in by tomorrow Wed @ 6:00 p.m.
(slide under office door LCB 204 if I'm not there.)

Warm-up Exercise: no: Any HW questions?

your solns should be

14.1 a)



$\frac{a_0}{2} = \frac{1}{6}$

$a_n = \frac{1}{n\pi}\sin\left(\frac{n\pi}{3}\right)$

$b_n = -\frac{1}{n\pi}\left(\cos\frac{n\pi}{3} - 1\right)$

if you wanted piecewise def (which is not rec.)

$n = 6k$
$6k+1$
$6k+2$
$6k+3$
$6k+4$
$6k+5$

since $\sin\left(\frac{n\pi}{3}\right)$ has period $n = 6$.

$2L = 6\pi$
$L = 3\pi$

$$f = \frac{1}{6} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{nt}{3}\right)$$

$\frac{\pi n}{L} = \frac{\cancel{\pi} n}{3\cancel{\pi}}$

$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos\frac{n\pi}{3} - 1\right)\sin\left(\frac{nt}{3}\right)$$

-2

$x'' + x = f(t)$

$w_0 = 1$; do any w's in f's Fourier exp $= w_0 = 1$

$n = 3$ terms but $\sin\left(\frac{3\pi}{3}\right) = 0$
not there.
$\sin t$ term.

Odds and ends, for fun: The non Fourier series ("d'Alembert") approach to the wave equation: For

$$u_{tt} = a^2 u_{xx} \qquad -\infty < x < \infty$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

the solution is

$$u(x,t) = \frac{1}{2}\left(f(x-at) + f(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\, ds$$

$$= \frac{1}{2}\left(f(x-at) + f(x+at)\right) + \frac{1}{2a}\left(G(x+a\,t) - G(x-a\,t)\right).$$ G any antideriv.

Check! $u(x,0) = \frac{1}{2}\left(f(x-0) + f(x+0)\right) + \frac{1}{2a}\cancel{\int_x^x} \to 0 \quad = f(x)$ ✓

$u_t(x,t) = \frac{1}{2}\left[f'(x-at)(-a) + f'(x+at)a\right] + \frac{1}{2a}\left(G'(x+at)a + G'(x-at)a\right)$

@ $t=0$: $u_t(x,0) = \left[f'(x)\underbrace{(\cancel{-a+a})}_{0}\right] + \frac{1}{2a}\left[G'(x)(a+a)\right] = G'(x) = g(x)$

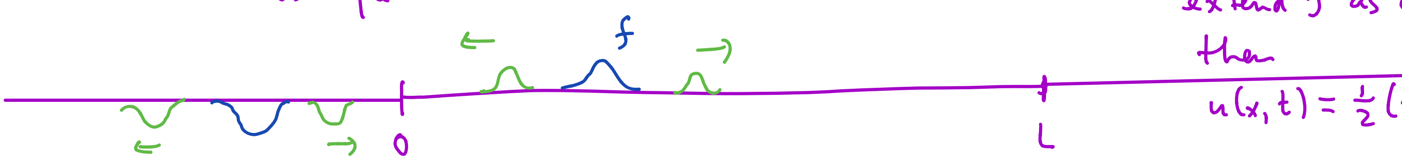
You can combine the fact above with even and odd extensions of initial data from the interval $0 < x < L$ to solve the natural initial boundary value problems we studied yesterday with Fourier series. We'll illustrate this fact for a slinky initial value problem in class.

$u(0,t) = u(L,t) = 0$

example. $u(x,0) = f(x)$, $u_t(x,0) = 0$.

extend f as odd

then



$u(x,t) = \frac{1}{2}\left(f(x-at) + f(x+at)\right)$

Final Exam Review
Math 2280-002
Spring 2019

3:30

The final exam is Tuesday April 30, 1:00 -3:00 p.m., in our classroom LCB 219. There will be a review session, tentatively scheduled for this Thursday April 25, from 6:00-8:00 p.m., with room ~~TBA~~. LCB 121
Most of that session will be devoted to going over a practice exam (which I will post on CANVAS by some time on Wednesday), but please bring any other questions you may have.

The exam will be comprehensive. Precisely, you can expect anything we covered from sections 1.1-1.5, 2.1-2.4 (not 2.5-2.6), 3.1-3.6, 4.1, 5.1-5.4, 5.6, 9.1-9.6. Exam material will be weighted towards topics which have not yet been tested, i.e. Chapter 5.4, 5.6, 9.1-9.5 material (which relies conceptually on earlier work). As on the midterms, I will test your mathematical modeling abilities, your understanding of the key concepts, as well as your ability to do computations. The exam is closed book and closed note, except for a single index card of size up to 6 by 8 inches which you may bring to the exam and fill with as much or as little information as you want. You may use a scientific calculator to check arithmetic, if you wish. I will provide you with the Fourier coefficient formulas as well as the integral tables from the textbook covers and a table of particular solutions from Chapter 3 applications.

Copies of my final exams from previous years can be found on my "old classes" web page, although the topics list has varied through the years. For example, in previous years there were always some questions about the Laplace transform (Chapter 7), which we did not cover this year. (We replaced it with Chapter 6 on systems of first order autonomous differential equations, and also covered more Fourier series topics than in recent incarnations of my Math 2280 classes.) This semester's midterms, quizzes, homework, class notes, and the text are all good references. It always worked well for me as a student to make my own course outline with the key ideas (which I would then make sure I could explain and work with).

The exam is closed book and closed note, except that you may bring one You may use a scientific (but not a graphing) calculator, although symbolic answers are accepted for all problems, so no calculator is really needed.

My estimate for topics weighting (multiple topics can be addressed in a single question)

Chapters:
1-2: 10-20% first order DEs and applications
3: 15-30% linear differential equations and applications
4.1, 5: 30-50% linear systems of differential equations and applications
6: 15-20% autonomous first order systems of differential equations
9.1-9.6: 15-25% Fourier series and applications to forced oscillations and partial differential equations

from our syllabus - let's compare

Learning objectives for Math 2280: The goal of Math 2280 is to master the basic tools and problem solving techniques important in differential equations, as well as to understand the larger conceptual framework that ties these ideas together. Specific goals include:

- Be able to model dynamical systems that arise in math, science and engineering, by using general principles to derive the governing differential equations or systems of differential equations. These principles include linearization, compartmental analysis, Newtons laws and conservation of energy.
- Learn solution techniques for first order separable and linear differential equations. Solve initial value problems in these cases, with applications to problems in science and engineering.
- Understand how to approximate solutions even when exact formulas do not exist. Visualize solution graphs and numerical approximations to initial value problems via slope fields. Understand phase diagram analysis for autonomous first order differential equations.
- Apply vector space concepts from linear algebra such as linear combinations, span, independence, basis and dimension, to understand the solution space to linear differential equations and linear systems of differential equations.
- Learn how to solve constant coefficient linear differential equations via superposition, particular solutions, and homogeneous solutions found via characteristic equation analysis. Apply these techniques to understand solutions to the basic unforced and forced mechanical and electrical oscillation problems.
- ~~Learn how to use Laplace transform techniques to solve linear differential equations, with an emphasis on the initial value problems of mechanical systems, electrical circuits, and related problems.~~
- Understand the natural initial value problems for first order systems of differential equations, and how they encompass the natural initial value problems for higher order differential equations and general systems of differential equations.
- Be able to apply matrix algebra concepts related to eigenvalues, eigenvectors and matrix diagonalization, in order to find the general solution space to first and second order constant coefficient homogeneous linear systems of differential equations, especially those arising from compartmental analysis and mechanical systems.
- Learn how to work with matrix exponentials and other fundamental matrix solutions, as tools for understanding linear systems of differential equations with constant coefficients.
- Understand and be able to use linearization as a technique to understand the behavior of nonlinear dynamical systems near equilibrium solutions. Apply these techniques to autonomous systems of two first order differential equations, including interacting populations and systems arising from non-linear forced oscillation problems.
- Learn about Fourier series, and use them as an "infinite superposition" tool to study forced oscillation problems.
- Learn how to find solutions to ~~Laplace's equation~~, the heat equation and the wave equation using separation of variables, together with Fourier series and superposition.
- Develop your abilities to model dynamical systems with differential equations, and to understand solutions analytically and with technology and software such as Matlab and internet-based tools as appropriate.

Chapters 1-2:

- 1.1: basic terminology and meanings for differential equations and solution to differential equations and initial value problems

- 1.2, 1.4, 1.5: antidifferentiation differential equations, separable differential equations, linear differential equations
  - how to recognize and solve.
  - applications to input-output problems, Newton's law of cooling, exponential growth/decay,etc.
- 1.3: existence and uniqueness for solutions to first order IVP's, connection to slope field intuition.
- 2.1: improved population models
  - models: logistic, doomsday-extinction, harvesting
  - partial fractions and separation of variables to solve.
- 2.3: improved velocity and acceleration models.
  - linear or quadratic drag, modeling and solving IVP's.
- 2.2: autonomous first order differential equations
  - phase diagram analysis
  - using phase diagrams to understand long-time behavior
- 2.4: Euler's method: be able to one or two steps and relate to slope field.

Chapter 3)

- 3.1-3.2: second order linear differential equations and vector space theory for homogeneous and non-homogeneous solutions , i.e. why solutions $y = y_P + y_H$ for linear transformation equations $L(y) = f$, and why solutions to $L(y) = 0$ form a vector space.
- 3.3: characteristic polynomial and using roots to find homogeneous solution space, all cases.
- 3.4: applications to unforced mass-spring or pendulum problems
  - the mass-spring model, the pendulum model for small oscillations
  - undamped, underdamped, critically damped, over damped behavior and solutions
- 3.5: Finding particular solutions y_p to solve $L(y) = f$, then using the complete solution $y = y_p + y_H$ to solve initial value problems.
  - Undetermined coefficients either in math examples, or in mass-spring oscillation examples from 3.6.
- 3.6 Forced oscillation problems:
  - undamped phenomena: superposition with homogeneous solution, beating, resonance
  - damped phenomena: steady periodic and transient solutions; practical resonance. amplitude-phase form.
  - using conservation of energy TE=PE+KE to derive differential equations of motion for mass-spring and pendulum configuration.

Chapter 4) 4.1

- 4.1 Systems of differential equations
  - existence-uniqueness theorem for systems of first order DE's
  - how to convert a second order (or higher order) DE IVP to an equivalent first order system of DE's IVP, and the equivalences between the two frameworks.
  - modeling input-output systems (e.g. tanks) for solute amounts in each compartment

Chapter 5) 5.1-5.4, 5.6

- 5.1 Theory for linear systems of differential equations:
  - why the solution to $L(y) = f$ is $y = y_P + y_H$
  - why solution space to $L(y) = 0$ is a subspace, and what its dimension is (based on existence-uniqueness theorem).
  - Calculus differentiation rules for sums and products of vector and matrix valued functions.
- 5.2 Eigenvalue-eigenvector method for solving $\boldsymbol{x}'(t) = A\,\boldsymbol{x}$ (Most naturally coupled with an input-output problems or first order system versions of Chapter 3 mass-spring problems).
  - diagonalizable case with real eigenvalues or complex eigenvalues
  - solving $\boldsymbol{x}'(t) = A\,\boldsymbol{x} + \boldsymbol{f}(t)$ for simple $\boldsymbol{f}$, either with $\boldsymbol{x} = \boldsymbol{x}_P + \boldsymbol{x}_H$ or via change of functions using Math 2270 diagonalization identities.
- 5.3 reproducing and classifying phase portraits for $x' = A\,x$ when $n = 2$ using eigendata and general solutions to sketch phase portraits with real eigenvalues; using eigenvalues and sampling tangent field along coordinate axes to sketch portraits in case of complex eigenvalues.
- 5.4 mass- spring systems ~~(after the second midterm)~~, $\underline{\boldsymbol{x}}'' = A\,\underline{\boldsymbol{x}}$, ~~$\underline{\boldsymbol{x}}'' = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}$.~~
  - modeling longitudinal vibrations in mass-spring systems
  - Using eigendata to solve unforced mass spring systems $\underline{\boldsymbol{x}}'' = A\,\underline{\boldsymbol{x}}$
  - Identifying and interpreting the fundamental modes and their frequencies

for sure.

- 5.6 matrix exponentials and applications ~~(after the second midterm)~~
  - Power series definition; when $e^{A+B} = e^A e^B$; why $\left(e^A\right)^{-1} = e^{-A}$
  - How to compute e^{tA} using power series (if you're lucky), diagonalizability, a fundamental matrix solution, or for nilpotent matrices, or related combinations.
  - Using matrix exponentials to solve homogeneous first order system IVPs.
  - Deriving and using the matrix exponential solution formula to the IVP

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t) \qquad \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0.$$

after 2nd midterm.

Something, for sure

Chapter 6) 6.1-6.4

- 6.1-6.2 Identifying equilibria of first order systems of two autonomous differential equations algebraically. Using linearization and eigendata from Jacobian matrices to classify the type of equilibrium solution, understand the implications for stability, and to be able to sketch what the phase portrait looks like near the equilibrium solution. Interpreting pplane phase portraits.
- 6.3 population models, and what the various terms in the model represent.
- 6.4 nonlinear mechanical models, e.g pendulum and nonlinear springs. Using conservation of energy (or other conserved quantities e.g in section 6.3) in cases where linearization near an equilibrium point is indeterminant, in order to deduce a stable center for the nonlinear problem.

Chapter 9) 9.1-9.6 Fourier series and applications (after second midterm)

I will keep computations easy

- 9.1 Fourier series for $2\pi-$ periodic functions.
  - how to compute
  - convergence theorems
  - inner product space projection, with orthogonal basis interpretation of truncated Fourier series.
- 9.2 Fourier series for $2L-$ periodic functions.
  - how to compute
  - using Fourier series for $2\pi-$ periodic functions and variable scaling to construct ones for $2L-$ periodic functions
  - differentiating and integrating Fourier series term by term
- 9.3 even and odd extensions of function on the interval $[0, L]$ and how to compute the corresponding cosine and sine series.
- 9.4 understanding resonance and practical resonance for forced oscillation problems with periodic forcing, via Fourier series.
  - superposition of particular solutions for the Fourier expansion of the forcing function, (Table will be provided if such a problem appears), and identifying key terms.

have asked in past

- 9.5 Heat equation
  - the model
  - the natural initial boundary value problems
  - the natural product solutions, and using these and superposition to solve the natural initial boundary value problems
- 9.6 Wave equation
  - the model
  - the natural initial boundary value problems
  - the natural product solutions, and using these and superposition to solve the natural initial boundary value problems

like simpler HW problems

key fact:
from products solns
$\cos\left(\frac{n\pi}{L}x\right)\ v(t)$
$\sin\left(\frac{n\pi}{L}x\right)\ v(t)$
and superposition you can solve the natural initial boundary value problems.