Math 2280-001
Week 7: 3.5-3.6
Wed Feb 22

Section 3.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems, and because sometimes a good choice for y_P contains the most essential information in dynamical systems problems).

There are two methods we will use:

• The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 3.5.

• The method of variation of parameters is more general, and yields an integral formula for a particular solution y_P, assuming you are already in possession of a basis for the homogeneous solution space. This method has the advantage that it works for any linear differential equation and any (continuous) function f. It has the disadvantage that the formulas can get computationally messy especially for differential equations of order $n > 2$.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise later in today's notes.

Exercise 1) Find a particular solution $y_P(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 10x + 3\ .$$

Hint: try $y_P(x) = d_1 x + d_2$ because L transforms such functions into ones of the form $b_1 x + b_2$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 10, b_2 = 3$.

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4\,y' - 5\,y = 10\,x + 3$$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4\,y' - 5\,y = 14\,e^{2\,x}\,.$$

Hint: try $y_P = d\,e^{2\,x}$ because L transforms functions of that form into ones of the form $b\,e^{2\,x}$, i.e. $L\left(d\,e^{2\,x}\right) = b\,e^{2\,x}$. "$d$" is your "undetermined coefficient" for $b = 14$.

Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6\,.$$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6$$
$$y(0) = 4$$
$$y'(0) = -4\,.$$

4c) Check your answer with technology.

```
> with(DEtools) :
> dsolve( {y''(x) + 4·y'(x) − 5·y(x) = 14·e^(2·x) − 20·x − 6, y(0) = 4, y'(0) = -4});
```

$$y(x) = \frac{8}{5}\,\mathrm{e}^{-5\,x} - 4\,\mathrm{e}^{x} + 2\,\mathrm{e}^{2\,x} + 4\,x + \frac{22}{5} \qquad (1)$$

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>
```

Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = 2\cos(3x) .$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \to V$.

- In Exercise 1 $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2 x$.
- In Exercise 3 $V = span\{e^{2x}\}$ and so we guessed $y_P = d e^{2x}$.
- What's the smallest subspace V we can take in the current exercise? Can you see why

$V = span\{\cos(3x)\}$ and a guess of $y_P = d\cos(3x)$ won't work?

All of the previous exercises rely on:

Method of undetermined coefficients (base case): Let $L : V \to V$ be a linear transformation, with V a finite dimensional vector space, and let $f \in V$. Then $\exists!\ y_P \in V$ with $L\left(y_P\right) = f$ if and only if the only $y \in V$ for which $L(y) = 0$ is $y = 0$.

why: You definitely learned this fact in Math 2270, for the special case of matrix transformations $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\underline{\boldsymbol{x}}) = A_{n\times n}\underline{\boldsymbol{x}}$. (Each non-homogeneous matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution $\underline{\boldsymbol{x}}$ if and only if A reduces to the identity matrix I, if and only if the only solution to the homogeneous equation $\underline{\boldsymbol{A}}\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ is $\underline{\boldsymbol{x}} = \underline{0}$.) The theorem above is a generalization of this fact to general linear transformations $L : V \to V$. There is an "appendix" explaining the reasoning at the end of today's notes. In fact, it's a special case of the "rank+nullity" theorem, which you may have learned in Math 2270, and which you will review in a homework problem.

http://en.wikipedia.org/wiki/Rank–nullity_theorem

Exercise 6) Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

In other words, *specify* a finite-dimensional subpace V with $L : V \rightarrow V$ and with the only $y \in V$ satisfying $L(y) = 0$ being the zero function $y \equiv 0$.

6a) $L(y) = x^3 + 6x - 5$

6b) $L(y) = 4 e^{2x} \sin(3x)$

6c) $L(y) = x \cos(2x)$

BUT LOOK OUT

Exercise 7a) Find a particular solution to

$$y'' + 4y' - 5y = 4e^x .$$

Hint: since $p(r) = r^2 + 4r - 5 = (r-1)(r+5)$ and $y_H = c_1 e^x + c_2 e^{-5x}$, a guess of $y_P = a\, e^x$ will not work (and $span\{e^x\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_P = d\, x\, e^x$$

and factor $L = \mathrm{D}^2 + 4\,\mathrm{D} - 5\,I = (\mathrm{D} + 5\,I) \circ (\mathrm{D} - I)$.

7b) check work with technology

```
> with(DEtools) :
> dsolve(y''(x) + 4·y'(x) − 5·y(x) = 4·e^x, y(x));
>
```

A vector space theorem like the one for the base case, except for $L : V \rightarrow W$, combined with our understanding of how to factor constant coefficient differential operators (as in last week's homework) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in section 3.5 of the text, pages 190-191 of the new edition of our text, and the table 3.5.1, reproduced here.

Method of undetermined coefficients (extended case): If L has a factor $(\mathrm{D} - r\, I)^s$ and $\mathrm{e}^{r\,x}$ is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s , as in Exercise 7. (If you understood the homework problem last week about factoring L into composition of terms like $(\mathrm{D} - r\, I)^s$, then you have an inkling of why this recipe works. If you didn't understand that last week problem, there's another one this week so you get a second chance. :-)) You may also need to use superposition, as in Exercise 4, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	y_P	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1 + \ldots + b_m x^m$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)$	$r = 0$
$b_1 \cos(\omega\, x) + b_2 \sin(\omega\, x)$	$x^s\left(c_1 \cos(\omega\, x) + c_2 \sin(\omega\, x)\right)$	$r = \pm\, i\, \omega$
$\mathrm{e}^{a\,x}\left(b_1 \cos(\omega\, x) + b_2 \sin(\omega\, x)\right)$	$x^s \mathrm{e}^{a\,x}\left(c_1 \cos(\omega\, x) + c_2 \sin(\omega\, x)\right)$	$r = a \pm i\omega$
$b_0 \mathrm{e}^{a\,x}$	$x^s c_0 \mathrm{e}^{a\,x}$	$r = a$
$\left(b_0 + b_1 + \ldots + b_m x^m\right)\mathrm{e}^{a\,x}$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)\mathrm{e}^{a\,x}$	$r = a$

Exercise 8) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for y_P. Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

8a) $y''' + 2y'' = x^2 + 6x$

(So the characteristic polynomial for $L(y) = 0$ is $r^3 + 2r^2 = r^2(r+2) = (r-0)^2(r+2)$.)

> $dsolve\left(y'''(x) + 2\cdot y''(x) = x^2 + 6\cdot x, y(x)\right);$

$$y(x) = \frac{1}{24}x^4 + \frac{5}{12}x^3 + \frac{1}{4}e^{-2x}\_C1 - \frac{5}{8}x^2 + \_C2\,x + \_C3 \qquad \textbf{(2)}$$

>

8b) $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$

(So the characteristic polynomial for $L(y) = 0$ is

$r^2 - 4r + 13) = (r-2)^2 + 9 = (r-2+3i)(r-2-3i)$.)

> $dsolve\left(y''(x) - 4\cdot y'(x) + 13\cdot y(x) = 4\cdot e^{2\cdot x}\cdot\sin(3\cdot x), y(x)\right);$

$$y(x) = e^{2x}\sin(3x)\,\_C2 + e^{2x}\cos(3x)\,\_C1 - \frac{2}{3}e^{2x}\cos(3x)\,x \qquad \textbf{(3)}$$

>

Not every right hand side is amenable to finding particular solutions via undetermined coefficients. Luckily there is a more general (but technically messier) way that will always work:

Variation of Parameters: The advantage of this method is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L, and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x), \ldots y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0 .$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x), \ldots u_n(x)$ have derivatives satisfying the Wronskian matrix equation

$$\begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}$$

Here's how to check this fact when $n = 2$: Write

$$y_p = y = u_1y_1 + u_2y_2 .$$

Thus

$$y' = u_1y_1' + u_2y_2' + \left(u_1'y_1 + u_2'y_2\right) .$$

Set

$$\left(u_1'y_1 + u_2'y_2\right) = 0.$$

Then

$$y'' = u_1y_1'' + u_2y_2'' + \left(u_1'y_1' + u_2'y_2'\right) .$$

Set

$$\left(u_1'y_1' + u_2'y_2'\right) = f.$$

Notice that the two $(\ldots)$ equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the $n = 2$ version of the claimed condition for y_p. Under these conditions we compute

$$\begin{aligned} & p_0\left[y = u_1y_1 + u_2y_2\right] \\ & + p_1\left[y' = u_1y_1' + u_2y_2'\right] \\ & + 1\left[y'' = u_1y_1'' + u_2y_2'' + f\right] \\ & L(y) = u_1L\left(y_1\right) + u_2L\left(y_2\right) + f \\ & L(y) = 0 + 0 + f = f \end{aligned}$$

Exercise 9) Rework Exercise 7a with variation of parameters, i.e. find a particular solution to

$$y'' + 4\,y' - 5\,y = 4\,e^{x}$$

of the form

$$y_P(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = u_1 e^{x} + u_2 e^{-5\,x}.$$

Appendix: The following two theorems justify the method of undetermined coefficients, in both the "base case" and the "extended case." There is also a related homework problem.

Theorem 0:

- Let V and W be vector spaces. Let V have dimension $n < \infty$ and let $\{y_1, y_2, ..., y_n\}$ be a basis for V.
- Let $L : V \rightarrow W$ be a linear transformation, i.e. $L(y+z) = L(y) + L(z)$ and $L(c\,y) = c\,L(y)$ holds $\forall\ y, z \in V, c \in \mathbb{R}$.) Consider the range of L, i.e.

$$\begin{aligned} Range(L) &:= \{L(d_1 y_1 + d_2 y_2 + ... + d_n y_n) \in W, \textit{such that each } d_j \in \mathbb{R}\} \\ &= \{d_1 L(y_1) + d_2 L(y_2) + ... + d_n L(y_n) \in W, \textit{such that each } d_j \in \mathbb{R}\} \\ &= span\{L(y_1), L(y_2), ... L(y_n)\}. \end{aligned}$$

Then $Range(L)$ is $n - dimensional$ if and only if the only solution to $L(y) = 0$ is $y = 0$.

proof:

(i) $\Leftarrow$: The only solution to $L(y) = 0$ is $y = 0$ implies $Range(L)$ is $n - dimensional$:

If we can show $L(y_1), L(y_2), ... L(y_n)$ are linearly independent, then they will be a basis for $Range(L)$ and thus this subspace will have dimension n. So, consider the dependency equation:

$$d_1 L(y_1) + d_2 L(y_2) + ... + d_n L(y_n) = 0 .$$

Because L is a linear transformation, we can rewrite this equation as

$$L(d_1 y_1 + d_2 y_2 + ... + d_n y_n) = 0.$$

Under our assumption that the only homogeneous solution is the zero vector, we deduce

$$d_1 y_1 + d_2 y_2 + ... + d_n y_n = 0.$$

Since $y_1, y_2, ..., y_n$ are a basis they are linearly independent, so $d_1 = d_2 = ... = d_n = 0$.

□

(ii) $\Rightarrow$: $Range(L)$ is $n - dimensional$ implies the only solution to $L(y) = 0$ *is* $y = 0$: Since the range of L is $n - dimensional$, $L(y_1), L(y_2), ... L(y_n)$ must be linearly independent. Now, let $y = d_1 y_1 + d_2 y_2 + ... + d_n y_n$ be a homogeneous solution, $L(y) = 0$. In other words,

$$\begin{aligned} & L(d_1 y_1 + d_2 y_2 + ... + d_n y_n) = 0 \\ \Rightarrow\ & d_1 L(y_1) + d_2 L(y_2) + ... + d_n L(y_n) = 0 \\ \Rightarrow\ & d_1 = d_2 = ... = d_n = 0 \Rightarrow y = 0 . \end{aligned}$$

□

Theorem 1 Let Let V and W be vector spaces, both with the same dimension $n < \infty$. Let $L : V \rightarrow W$ be a linear transformation. Let the only solution to $L(y) = 0$ be $y = 0$. Then for each $f \in W$ there is a unique $y \in V$ with $L(y) = f$.

proof: By Theorem 0, the dimension of $Range(L)$ is $n - dimensional$. Therefore it must be all of W. So for each $f \in W$ there is at least one $y_P \in V$ with $L(y_P) = f$. But the general solution to $L(y) = f$ is $y = v_P + y_H$, where y_H is the general solution to the homogeneous equation. By assumption, $y_H = 0$, so the particular solution is unique.

□

Remark: In the base case of undetermined coefficients, $W = V$. In the extended case, W is the space in which f lies, and $V = x^s W$, i.e. the space of all functions which are obtained from ones in W by multiplying them by x^s. This is because if L factors as

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \ldots \circ (D - r_m I)^{k_m}$$

and if f is in a subspace W associated with the characteristic polynomial root r_m, then for $s = k_m$ the factor $(D - r_m I)^{k_m}$ of L will transform the space $V = x^s W$ back into W, and not transform any non-zero function in V into the zero function. And the other factors of L will then preserve W, also without transforming any non-zero elements to the zero function.

Math 2280-001
Fri Feb 24

Section 3.6: forced oscillations in mechanical systems (and as we shall see in section 3.7, also in electrical circuits) overview:
We study solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)$$

using section 3.5 undetermined coefficients algorithms.

- undamped $(c = 0)$:
  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\,x = 0$$

which has simple harmonic motion solutions

$$x_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = C_0 \cos(\omega_0 t - \alpha)\,.$$

So for the non-homongeneous DE the section 5.5 method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}}$ $\Rightarrow x_P = A \cos(\omega\,t)$ because only even derivatives, we don't need $\sin(\omega\,t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega\,t) + C_0 \cos(\omega_0 t - \alpha_0)\,.$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $A \approx C_0$ Beating!
- $\omega = \omega_0$ case 2 section 3.5 undetermined coefficients; since

$$p(r) = r^2 + \omega_0^2 = (r + i\omega_0)^1 (r - i\omega_0)^1$$

our undetermined coefficients guess is

$$x_P = t^1\left(A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\right)$$
$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega\,t - \alpha) + C_0\cos(\omega_0 t - \alpha_0)\,.$$

("pure" resonance!)

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$ (because the roots of the characteristic polynomial are never purely imagninary $\pm i\,\omega$ when $c > 0$).
  - underdamped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)\,.$
  - critically-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}(c_1 t + c_2)\,.$
  - over-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\,.$

- in all three cases on the previous page, $x_H(t) \to 0$ exponentially and is called the transient solution $x_{tr}(t)$ (because it disappears as $t \to \infty$).

$x_P(t)$ as above is called the steady periodic solution $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit practical resonance. This can be an important phenomenon in electrical circuits, where amplifying signals is important. We don't generally want pure resonance or practical resonance in mechanical configurations.

Forced undamped oscillations:

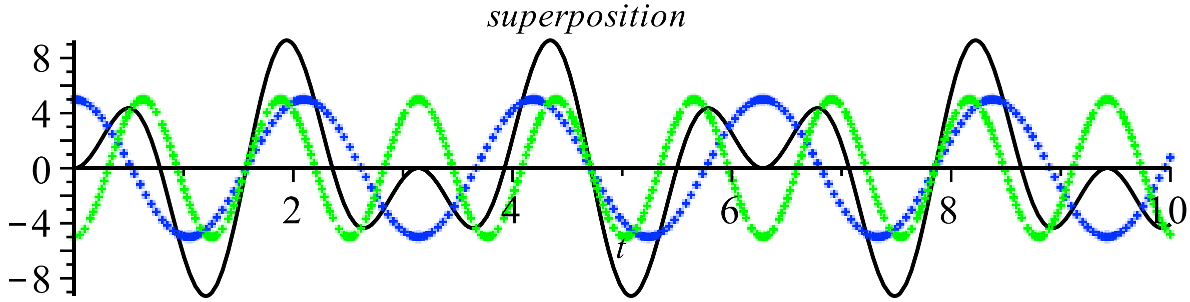
Exercise 1a) Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

1b) This superposition of two sinusoidal functions is periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

```
> with(plots):
> plot1 := plot(-5·cos(5·t), t = 0..10, color = green, style = point):
  plot2 := plot(5·cos(3·t), t = 0..10, color = blue, style = point):
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0..10, color = black):
  display({plot1, plot2, plot3}, title='superposition');
```



In general:

undamped forced IVP, $\omega \neq \omega_0$, with letters

$$\begin{cases} x'' + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\begin{aligned}
+\tfrac{k}{m}\,(x_p &= A\cos\omega t \ ) \\
+0\,(x_p' &= -A\omega\sin\omega t \ ) \\
+1\,(x_p'' &= -A\omega^2\cos\omega t \ ) \\
\hline
L(x_p) &= \cos\omega t\, A\,[\underbrace{\tfrac{k}{m}}_{\omega_0^2}-\omega^2]
\end{aligned}$$

deduce $A(\omega_0^2-\omega^2) = \frac{F_0}{m}$

$$A = \frac{F_0}{m} \Big/ (\omega_0^2 - \omega^2)$$

So,
$x_p(t) = -\frac{F_0}{m}\frac{1}{(\omega^2-\omega_0^2)}\cos\omega t$. Note $x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t$.

so, by plugging in or observation IVP solution is

$$\boxed{x(t) = \frac{+F_0}{m(\omega^2-\omega_0^2)}\left(\cos\omega_0 t - \cos\omega t\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

check · WR!

There is an interesting <u>beating</u> phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned}
\cos(\alpha-\beta) - \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\
&\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) \\
&= 2\sin(\alpha)\sin(\beta) \ .
\end{aligned}$$

Set $\alpha = \frac{1}{2}(\omega+\omega_0)t$, $\beta = \frac{1}{2}(\omega-\omega_0)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m(\omega^2-\omega_0^2)}2\sin\left(\frac{1}{2}(\omega+\omega_0)t\right)\sin\left(\frac{1}{2}(\omega-\omega_0)t\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t) \ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}(\omega-\omega_0), \qquad \text{period: } \frac{4\pi}{|\omega-\omega_0|} \ .$$

We will call half that period the beating period, as explained by the next exercise:

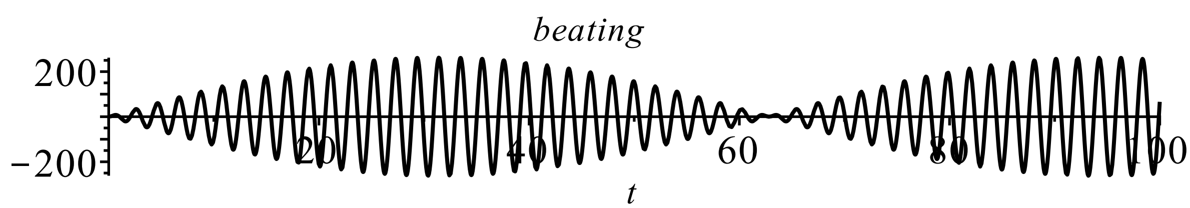
$$\text{beating period: } \frac{2\pi}{|\omega - \omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m|\omega^2 - \omega_0^2|}.$$

Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\,t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

> $plot(262.3 \cdot \sin(3.05 \cdot t)\sin(.05 \cdot t),\ t = 0..100,\ color = black,\ title = \text{`beating`});$



Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned}
+\omega_0^2 (\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \quad)\\
0 (\quad & x_p' = t\,(-A\omega_0 \sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \quad)\\
+1 (\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \quad)
\end{aligned}$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\,t\sin\omega_0 t}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\,t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

(You can also get this solution by letting $\omega \to \omega_0$ in the beating formula.)

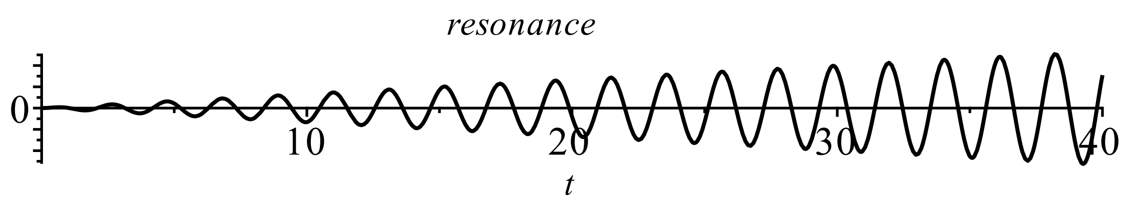
Exercise 3a) Solve the IVP

$$x'' + 9x = 80\cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

Either use the general solution formula above this exercise and substitute in the appropriate values for the various terms, or use this as a chance to practice variation of parameters for the particular solution.

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3} t \cdot \sin(3 \cdot t),\ t = 0..40,\ color = black,\ title = \text{`resonance`}\right);$



>

Damped forced oscillations $(c > 0)$ for $x(t)$:

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_P(t)$:

$$\begin{aligned} k\,&\left[x_P = A\cos(\omega t) + B\sin(\omega t)\right] \\ +\,c\,&\left[x_P' = -A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t)\right] \\ +\,m\,&\left[x_P'' = -A\,\omega^2\cos(\omega t) - B\,\omega^2\sin(\omega t)\right]\ . \end{aligned}$$

$$\begin{aligned} L(x_P) &= \cos(\omega t)\left(k\,A + c\,B\,\omega - m\,A\,\omega^2\right) \\ &\quad + \sin(\omega t)\left(k\,B - c\,A\,\omega - m\,B\,\omega^2\right)\ . \end{aligned}$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\,\omega^2 & c\,\omega \\ -c\,\omega & k - m\,\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}\begin{bmatrix} k - m\,\omega^2 & -c\,\omega \\ c\,\omega & k - m\,\omega^2 \end{bmatrix}\begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}\begin{bmatrix} k - m\,\omega^2 \\ c\,\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}\ . \quad \text{(Check!)}$$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}\ .$$

And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)$.
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t}(c_1 t + c_2)$.
- over-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

Important to note:

- The amplitude C in x_{sp} can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle α is always in the first or second quadrant.

Exercise 4) (a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice that our math on the previous page exactly predicts when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient c! Namely, for c small, when $\omega^2 << \omega_0^2$ we have α near zero (in phase) for x_{sp},because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx 1$; when $\omega^2 >> \omega_0^2$ we have α near π (out of phase), because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx -1$; for $\omega \approx \omega_0$, α is near $\frac{\pi}{2}$, because $\sin(\alpha) \approx 1$, $\cos(\alpha) \approx 0$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

Exercise 5) Solve the IVP for $x(t)$:

$$x'' + 2\,x' + 26\,x = 82\cos(4\,t)$$
$$x(0) = 6$$
$$x'(0) = 0\,.$$

Solution:

$$x(t) = \sqrt{41}\cos(4\,t - \alpha) + \sqrt{10}\,e^{-t}\cos(5\,t - \beta)$$
$$\alpha = \arctan(0.8),\ \beta = \arctan(-3)\,.$$

```
> with(DEtools) :
> dsolve({x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});
```

$$x(t) = -3\,e^{-t}\sin(5\,t) + e^{-t}\cos(5\,t) + 5\cos(4\,t) + 4\sin(4\,t) \qquad \textbf{(4)}$$

Practical resonance: The steady periodic amplitude C for damped forced oscillations (two pages back) is

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}\,.$$

Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$.

(Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in ω^2 it will have at most one minimum in the variable ω^2, so $C(\omega)$ will have at most one maximum for non-negative ω. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 6a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c:

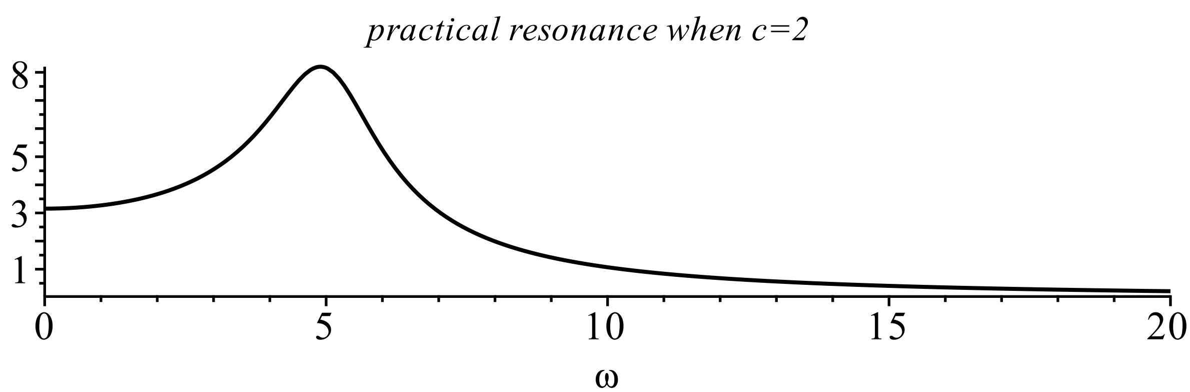
$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t).$$

6b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

> $with(plots):$

> $C := (\omega, c) \to \dfrac{82}{\sqrt{\left(26 - \omega^2\right)^2 + c^2\cdot\omega^2}}:$

> $plot(C(\omega, 2), \omega = 0..20, color = black, title = \text{`practical resonance when c=2`});$



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