Math 2280-001
Week 5, Feb 6-10, 3.1-3.4

Mon Feb 6 3.1-3.2

• Finish Friday's notes and examples about second order linear differential equations, including Theorems 1,2,3 and the associated examples, if necessary. The following comments are for us to consider after we finish Exercise 6 there:

Although we don't have the tools yet to prove the existence-uniqueness result Theorem 2 (Friday for $n = 2$, today in general) we can use it to prove the dimension result Theorem 3 (both days). Here's how, for $n = 2$ (and this is really just an abstractified version of the example in Friday's Exercise 6):

Consider the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

on an interval I for which the hypotheses of the existence-uniqueness theorem hold.
Pick any $x_0 \in I$. Find solutions $y_1(x), y_2(x)$ to IVP's at x_0 so that the so-called Wronskian matrix for y_1, y_2 at x_0

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible (i.e. $\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}$ are a basis for $\mathbb{R}^2$, or equivalently so that the determinant of the Wronskian matrix (called just the Wronskian) is non-zero at x_0).

• You may be able to find suitable y_1, y_2 by good guessing, as in the previous example, but the existence-uniqueness theorem guarantees they exist even if you don't know how to find formulas for them.

Under these conditions, the solutions y_1, y_2 are actually a basis for the solution space! Here's why:

• span: the condition that the Wronskian matrix is invertible at x_0 means we can solve each IVP there with a linear combination $y = c_1 y_1 + c_2 y_2$: In that case, $y' = c_1 y_1' + c_2 y_2'$ so to solve the IVP

$$y'' + p(x)y' + q(x)y = 0$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

we set

$$c_1 y_1(x_0) + c_2 y_2(x_0) = b_0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = b_1$$

which has unique solution $[c_1, c_2]^T$ given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Since the uniqueness theorem says each IVP has a unique solution, this must be it. Since each solution $y(x)$ to the differential equation solves *some* initial value problem at x_0, each solution $y(x)$ *is* a linear combination of y_1, y_2. Thus y_1, y_2 span the solution space.

• Linear independence: The computation above shows that there is only one way to write any solution $y(x)$ to the differential equation as a linear combination of y_1, y_2, because the linear combination coefficients c_1, c_2 are uniquely determined by the values of $y(x_0), y'(x_0)$. (In particular they must be zero if $y(x) \equiv 0$, because for the zero function b_0, b_1 are both zero so c_1, c_2 are too. This shows linear independence.)

3.2: general theory for n^{th}-order linear differential equations; tests for linear independence;
also begin 3.3: finding the solution space to homogeneous linear constant coefficient differential equations by trying exponential functions as potential basis functions.

The two main goals in Chapter 3 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = f$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$y''(x_0) = b_2$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

and to learn important physics/engineering applications of these general techniques.

The algorithm for solving these DEs and IVPs is:

(1) Find a basis $y_1, y_2, \dots y_n$ for the n-dimensional homogeneous solution space, so that the general homogeneous solution is their span, i.e. $y_H = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$.
(2) If the DE is non-homogeneous, find a particular solution y_P. Then the general solution to the non-homogeneous DE is $y = y_P + y_H$. (If the DE is homogeneous you can think of taking $y_P = 0$, since $y = y_H$.)
(3) Find values for the n free parameters $c_1, c_2, \dots c_n$ in

$$y = y_P + c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

to solve the initial value problem with initial values $b_0, b_1, \dots, b_{n-1}$. (This last step just reduces to a matrix problem, where the matrix is the Wronskian matrix of $y_1, y_2, \dots y_n$, evaluated at x_0 and the right hand side vector comes from the initial values and the particular solution and its derivatives' values at x_0.)

We've already been exploring how these steps play out in examples and homework problems, but will be studying them more systematically on Wednesday and Friday. On Friday we'll begin the applications in section 3.4. We should have some fun experiments next week to compare our mathematical modeling with physical reality.

Definition: An n^{th} order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + ... + A_1(x)y' + A_0(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A_n(x)$ $\neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f.$$

($a_{n-1}, ... a_1, a_0, f$ are all functions of x, and the DE above means that equality holds for all value of x in the interval I.)

This DE is called linear because the operator L defined by

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

• *The proof that L satisfies the linearity proporties is just the same as it was for the case when n = 2, that we checked Friday. Then, since the* $y = y_P + y_H$ *proof only depended on the linearity properties of L, we deduce both of Theorems 0 and 1:*

Theorem 0: The solution space to the homogeneous linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

is a subspace.

Theorem 1: The general solution to the nonhomogeneous n^{th} order linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f$$

is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

Later in the course we'll understand n^{th} order existence uniqueness theorems for initial value problems, in a way analogous to how we understood the first order theorem using slope fields, but let's postpone that discussion and just record the following true theorem as a fact:

Theorem 2 (Existence-Uniqueness Theorem): Let $a_{n-1}(x), a_{n-2}(x),\ldots a_1(x), a_0(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$\begin{aligned} y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y &= f \\ y(x_0) &= b_0 \\ y'(x_0) &= b_1 \\ y''(x_0) &= b_2 \\ &\vdots \\ y^{(n-1)}(x_0) &= b_{n-1} \end{aligned}$$

and $y(x)$ exists and is n times continuously differentiable on the entire interval I.

Just as for the case n = 2, the existence-uniqueness theorem lets you figure out the dimension of the solution space to homogeneous linear differential equations. The proof is conceptually the same, but messier to write down because the vectors and matrices are bigger.

Theorem 3: The solution space to the n^{th} order homogeneous linear differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y \equiv 0$$

is n-dimensional. Thus, any n independent solutions $y_1, y_2, \ldots y_n$ will be a basis, and all homogeneous solutions will be uniquely expressible as linear combinations

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

proof: By the existence half of Theorem 2, we know that there are solutions for each possible initial value problem for this (homogenous case) of the IVP for n^{th} order linear DEs. So, pick solutions $y_1(x), y_2(x), \ldots. y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ y_1''(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \\ y_2''(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix}, \ldots, \begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ y_n''(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}$$

are a basis for $\mathbb{R}^n$ (i.e. these n vectors are linearly independent and span $\mathbb{R}^n$. (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

Claim: In this case, the solutions $y_1, y_2, \ldots y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these n functions and the dimension of the solution space is n .... discussion on next page.

• Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the Wronskian matrix at x_0 times the vector of linear combination coefficients:

$$\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

has a unique solution $\underline{\mathbf{c}}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Thus $y_1, y_2, \ldots y_n$ <u>span</u> the solution space. If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then because the zero function has zero initial vector $[b_0, b_1, \ldots b_{n-1}]^T$ the matrix equation above implies that $[c_1, c_2, \ldots c_n]^T = \underline{\mathbf{0}}$, so $y_1, y_2, \ldots y_n$ are also <u>linearly independent</u>. Thus, $y_1, y_2, \ldots y_n$ are a <u>basis</u> for the solution space and the general solution to the homogeneous DE can be written as

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

□

Let's do some new exercises that tie these ideas together.

Exercise 1) Consider the 3^{rd} order linear homogeneous DE for $y(x)$:

$$y''' + 3y'' - y' - 3y = 0.$$

Find a basis for the 3-dimensional solution space, and the general solution. Use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

Exercise 2a) Find the general solution to

$$y''' + 3y'' - y' - 3y = 6 .$$

Hint: First try to find a particular solution ... try a constant function.

b) Set up the linear system to solve the initial value problem for this DE, with $y(0) = -1, y'(0) = 2, y''(0) = 7$.

for fun now, but maybe not just for fun later:

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> with(DEtools) :
  dsolve({y'''(x) + 3·y''(x) − y'(x) − 3·y(x) = 6, y(0) = −1, y'(0) = 2, y''(0) = 7});
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$$y(x) = -2 + \frac{9}{4}\mathrm{e}^{x} + \frac{3}{4}\mathrm{e}^{-3x} - 2\,\mathrm{e}^{-x} \qquad \textbf{(1)}$$

Math 2280-001
Wed Feb 8
3.2-3.3

- Review Monday's notes about the general theory for n^{th} order linear differential equations, section 3.2.

- In section 3.2 there is a focus on testing whether collections of functions are linearly independent or not. This is important for finding bases for the solution spaces to homogeneous linear DE's because of the fact that if we find n linearly independent solutions to the n^{th} order homogeneous DE, they will automatically span the $n-$dimensional the solution space. (Do you recall this linear algebra "magic fact", i. e. that n linearly independent vectors in an n-dimensional space automatically span the space and are therefore a basis? We can review it if you wish.) Checking just linear independence is sometimes easier than also checking the spanning property.

Ways to check whether functions $y_1, y_2, \ldots y_n$ are linearly independent on an interval:

In all cases you begin by writing the linear combination equation

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

where "0" is the zero function which equals 0 for all x on our interval.

Method 1) Plug in different $x-$ values to get a system of algebraic equations for $c_1, c_2 \ldots c_n$. Either you'll get enough "different" equations to conclude that $c_1 = c_2 = \ldots = c_n = 0$, or you'll find a likely dependency.

Exercise 1) Use method 1 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, \; y_2(x) = x, \; y_3(x) = x^2$$

are linearly independent. (These functions show up in the homework.) For example, try the system you get by plugging in $x = 0, -1, 1$ into the equation

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

Method 2) If your interval stretches to $+\infty$ or to $-\infty$ and your functions grow at different rates, you may be able to take limits (after dividing the dependency equation by appropriate functions of x), to deduce independence.

Exercise 2) Use method 2 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent. Hint: first divide the dependency equation by the fastest growing function, then let $x \to \infty$.

Method 3) If

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

$\forall\ x \in I$, then we can take derivatives to get a system

$$\begin{aligned} c_1 y_1 \quad + \quad c_2 y_2 + \ldots + \quad c_n y_n &= 0 \\ c_1 y_1{}' + c_2 y_2{}' + \ldots + c_n y_n{}' &= 0 \\ c_1 y_1{}'' + c_2 y_2{}'' + \ldots + c_n y_n{}'' &= 0 \end{aligned}$$

$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \ldots + c_n y_n^{(n-1)} = 0$$

(We could keep going, but stopping here gives us n equations in n unknowns.)
Plugging in any value of x_0 yields a homogeneous algebraic linear system of n equations in n unknowns, which is equivalent to the Wronskian matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If this Wronskian matrix is invertible at even a single point $x_0 \in I$, then the functions are linearly independent! (So if the determinant is zero at even a single point $x_0 \in I$, then the functions are independent....strangely, even if the determinant was zero for all $x \in I$, then it could still be true that the functions are independent....but that won't happen if our n functions are all solutions to the same n^{th} order linear homogeneous DE.)

Exercise 3) Use method 3 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent. Use $x_0 = 0$.

Remark 1) Method 3 is usually not the easiest way to prove independence in general. But we and the text like it when studying differential equations because as we've seen, the Wronskian matrix shows up when you're trying to solve initial value problems using

$$y = y_P + y_H = y_P + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

as the general solution to

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$

This is because, if the initial conditions for this inhomogeneous DE are

$$y(x_0) = b_0, y'(x_0) = b_1, \ldots y^{(n-1)}(x_0) = b_{n-1}$$

then you need to solve matrix algebra problem

$$\begin{bmatrix} y_P(x_0) \\ y_P'(x_0) \\ \vdots \\ y_P^{(n-1)}(x_0) \end{bmatrix} + \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

for the vector $[c_1, c_2, \ldots, c_n]^T$ of linear combination coefficients. And so if you're using the Wronskian matrix method, and the matrix is invertible at x_0 then you are effectively directly checking that $y_1, y_2, \ldots y_n$ are a basis for the homogeneous solution space, and because you've found the Wronskian matrix you are ready to solve any initial value problem you want by solving for the linear combination coefficients above.

Remark 2) There is a seemingly magic consequence in the situation above, in which $y_1, y_2, \ldots y_n$ are all solutions to the same n^{th}-order homogeneous DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

(even if the coefficients aren't constants): If the Wronskian matrix of your solutions $y_1, y_2, \ldots y_n$ is invertible at a single point x_0, then $y_1, y_2, \ldots y_n$ are a basis because linear combinations uniquely solve all IVP's at x_0. But since they're a basis, that also means that linear combinations of $y_1, y_2, \ldots y_n$ solve all IVP's at any other point x_1. This is only possible if the Wronskian matrix at x_1 also reduces to the identity matrix at x_1 and so is invertible there too. In other words, the Wronskian determinant will either be non-zero $\forall\, x \in I$, or zero $\forall\, x \in I$, when your functions $y_1, y_2, \ldots y_n$ all happen to be solutions to the same n^{th} order homogeneous linear DE as above.

Exercise 4) Verify that $y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$ all solve the third order linear homogeneous DE

$$y''' = 0,$$

and that their Wronskian determinant is indeed non-zero $\forall\, x \in \mathbb{R}$.

3.3: Algorithms for a basis and the general (homogeneous) solution to

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are all constant.

strategy: Try to find a basis made of exponential functions....try $y(x) = e^{r x}$. In this case

$$L(y) = e^{r x}\left(r^n + a_{n-1}r^{n-1} + ... + a_1 r + a_0\right) = e^{r x} p(r) .$$

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, \ldots, r_n$, then

$$e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$$

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x} .$$

Exercise 5) By construction, $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n-dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < \ldots < r_n$ and to use a limiting argument, as we did in Method 2 and Exercise 2 above.

Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, \ldots r_m$ $(m < n)$ with the r_j all different, but some of the factors $(r - r_j)$ in $p(r)$ appear with powers bigger than 1 . In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2}\ldots(r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \ldots + k_m = n$.

Start with a small example: The case of a second order DE for which the characteristic polynomial has a double root. There is an example like this in your section 3.1 homework for Friday.

Exercise 6) Let r_1 be any real number. Consider the homogeneous DE

$$L(y) := y'' - 2\,r_1\,y' + r_1^2\,y = 0\,.$$

with $p(r) = r^2 - 2\,r_1\,r + r_1^2 = (r - r_1)^2$, i.e. r_1 is a double root for $p(r)$. Show that $e^{r_1 x}, x\,e^{r_1 x}$ are a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is $y_H(x) = c_1 e^{r_1 x} + c_2\,x\,e^{r_1 x}$. Start by checking that $x\,e^{r_1 x}$ actually (magically?) solves the DE.

Here's the general algorithm for repeated real roots: If the characteristic polynomial

$$p(r) = (r - r_1)^{k_1} (r - r_2)^{k_2} \cdots (r - r_m)^{k_m},$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis for the n-dimensional solution space. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}.$$

This yields k_j solutions for each root r_j, so since $k_1 + k_2 + \ldots + k_m = n$ you get a total of n solutions to the differential equation. There's a good explanation as to why these additional functions actually do solve the differential equation, see page 165 and the discussion of "polynomial differential operators". I'll make a homework problem in your next assignment to explore these ideas, and discuss it a bit in class, on Friday. Using the limiting method we discussed earlier, it's not too hard to show that all n of these solutions are indeed linearly independent, so they are in fact a basis for the solution space to $L(y) = 0$.

Exercise 7) Explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in Case 2 above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H. When you compare to the repeated roots algorithm, note that it includes the possibility $r = 0$ and that $e^{0x} = 1$, $xe^{0x} = x$, etc.

Case 3) $p(r)$ has some complex roots. It turns out that exponential functions $e^{r x}$ still work, except that $r = a \pm b i$. However, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions. We will do the details carefully on Friday - they depend on Euler's (amazing) formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) .$$

However, the punchline (which you will use in your homework due this Friday), is that if $r = a \pm b i$ are two roots of the characteristic polynomial, then

$$y_1(x) = e^{a x}\cos(b x), y_2(x) = e^{a x}\sin(b x)$$

both solve the homogeneous differential equation!

More generally, Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, ..., a_1, a_0$. If $(r - (a + b i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b i))^k$. Associated to these two factors are $2k$ real and independent solutions to $L(y) = 0$, namely

$$\begin{array}{c} e^{a x}\cos(b x), e^{a x}\sin(b x) \\ x e^{a x}\cos(b x), x e^{a x}\sin(b x) \\ \vdots \quad\quad \vdots \\ x^{k-1} e^{a x}\cos(b x), x^{k-1} e^{a x}\sin(b x) \end{array}$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 8) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 9 y = 0 .$$

Exercise 9) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6 y' + 13 y = 0 .$$

Exercise 10) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r - 2)^3 .$$

What is the general solution to the corresponding homogeneous DE?

Math 2280-001
Fri Feb 10
3.3-3.4.

First, Leftovers from section 3.3:

Consider the homogeneous, constant coefficient linear differential equation for $y(x)$,

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y \equiv 0$$

and its characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0.$$

• Repeated roots mystery: Why, if $\left(r - r_j\right)^{k_j}$ is a factor of the characteristic polynomial $p(r)$, do we get k_j linearly independent solutions

$$e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, \ldots, x^{k_j - 1} e^{r_j x}\ ?$$

to the homogeneous DE? You will explore this mystery in your homework ...

• What to do when the characteristic polynomial has complex roots? We already discussed the fact that if $r = a \pm b\, i$ are complex roots of $p(r)$, then

$$y_1(x) = e^{a x} \cos(b x)$$
$$y_2(x) = e^{a x} \sin(b x)$$

are real-valued solutions to the DE. What is behind this mysterious fact? See following ...

(recall, Case 1 was distinct roots to characteristics poly; Case 2 was repeated roots)
Case 3) $p(r)$ has some complex roots. The punch line is that exponential functions $e^{r x}$ still work, except that $r = a \pm b\, i$. However, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions.

To understand how this all comes about, we need to recall or learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions are very important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

$$f(x) \sim f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + \ldots + \frac{1}{n!} f^{(n)}(0)x^n + \ldots$$

(Recall that the partial sum polynomial through order n matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also need error estimates to figure out on which intervals the Taylor polynomials coverge to f.)

Exercise 1) Use the formula above to recall the three very important Taylor series for

1a) $e^x =$
1b) $\cos(x) =$
1c) $\sin(x) =$

In Calculus you checked that these Taylor series actually converge and equal the given functions, for all real numbers *x*.

Exercise 2) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta)\ .$$

From Euler's formula it makes sense to define

$$e^{a + b\,i} := e^a e^{b\,i} = e^a\left(\cos(b) + i\sin(b)\right)$$

for $a, b \in \mathbb{R}$. So for $x \in \mathbb{R}$ we also get

$$e^{(a + b\,i)x} = e^{a\,x}\left(\cos(b\,x) + i\sin(b\,x)\right) = e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\ .$$

For a complex function $f(x) + i\,g(x)$ we define the derivative by

$$D_x\left(f(x) + i\,g(x)\right) := f'(x) + i\,g'(x)\ .$$

It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. The following rule pertains most specifically to our discussion and we should check it:

Exercise 3) Check that $D_x\left(e^{(a + b\,i)x}\right) = (a + b\,i)e^{(a + b\,i)x}$, i.e.

$$D_x\, e^{r\,x} = r\, e^{r\,x}$$

even if *r* is complex. (So also $D_x^2\, e^{r\,x} = D_x r\, e^{r\,x} = r^2\, e^{r\,x}$, $D_x^3\, e^{r\,x} = r^3\, e^{r\,x}$, etc.)

Now return to our differential equation questions, with

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y .$$

Then even for complex $r = a + b\,i$ $(a, b \in \mathbb{R})$, our work above shows that

$$L\left(e^{r x}\right) = e^{r x}\left(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0\right) = e^{r x}\, p(r) .$$

So if $r = a + b\,i$ is a complex root of $p(r)$ then $e^{r x}$ is a complex-valued function solution to $L(y) = 0$. But L is linear, and because of how we take derivatives of complex functions, we can compute in this case that

$$0 + 0\,i = L\left(e^{r x}\right) = L\left(e^{a x}\cos(b x) + i\, e^{a x}\sin(b x)\right)$$
$$= L\left(e^{a x}\cos(b x)\right) + i\, L\left(e^{a x}\sin(b x)\right) .$$

Equating the real and imaginary parts in the first expression to those in the final expression (because that's what it means for complex numbers to be equal) we deduce

$$0 = L\left(e^{a x}\cos(b x)\right)$$
$$0 = L\left(e^{a x}\sin(b x)\right) .$$

Upshot: If $r = a + b\,i$ is a complex root of the characteristic polynomial $p(r)$ then

$$y_1 = e^{a x}\cos(b x)$$
$$y_2 = e^{a x}\sin(b x)$$

are two solutions to $L(y) = 0$. (The conjugate root $a - b\,i$ would give rise to y_1, $-y_2$, which have the same span.

Case 3) Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$e^{a x}\cos(b x),\; e^{a x}\sin(b x)$$
$$x\, e^{a x}\cos(b x),\; x\, e^{a x}\sin(b x)$$
$$\vdots \qquad\qquad \vdots$$
$$x^{k-1}e^{a x}\cos(b x),\; x^{k-1}e^{a x}\sin(b x)$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 4) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

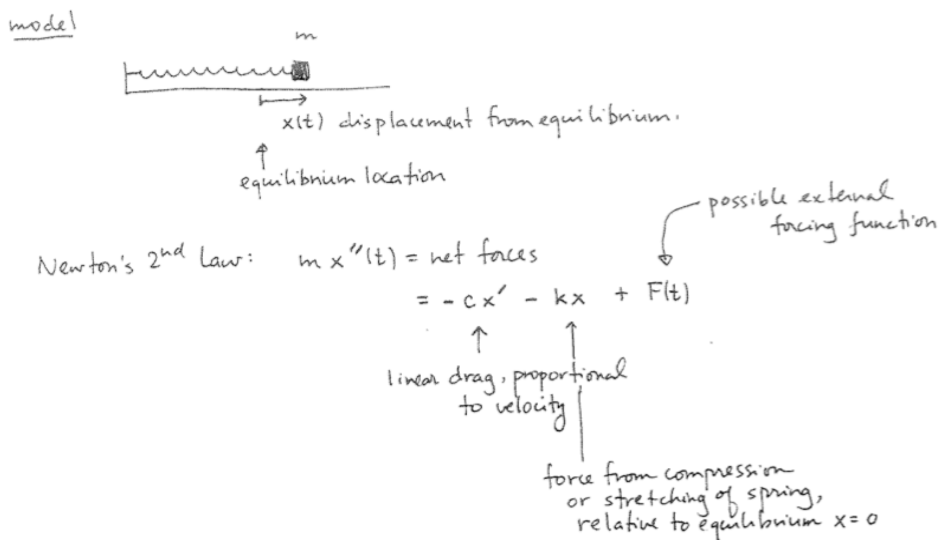
$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r - 2)^3 .$$

What is the general solution to the corresponding homogeneous DE?

3.4 applications of constant coefficient homogeneous linear differential equations to unforced mechanical oscillation problems.

In this section we study the differential equation below for functions $x(t)$ measuring the displacement of a mass from its equilibrium solution

$$m\,x'' + c\,x' + k\,x = 0\,.$$



In section 3.4 we assume the time dependent external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0,\ x' = 0$, for which the net forces must be zero (because the configuration stays at rest). Notice that $c \geq 0,\ k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t), x'(t)$ are sufficiently small. k is called the Hooke's constant, and c is called the damping coefficient.

This is a constant coefficient linear homogeneous DE, so we try $x(t) = e^{r\,t}$ and compute

$$L(x) := m\,x'' + c\,x' + k\,x = e^{r\,t}\left(m\,r^2 + c\,r + k\right) = e^{r\,t}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ pocesses...

Case 1) no damping ($c = 0$).

$$m\,x'' + k\,x = 0$$

$$x'' + \frac{k}{m}\,x = 0\,.$$

$$p(r) = r^2 + \frac{k}{m}\,,$$

has roots

$$r^2 = -\frac{k}{m} \quad \text{i.e.} \quad r = \pm\, i\sqrt{\frac{k}{m}}\,.$$

So the general solution is

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right).$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the natural angular frequency. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2} x = 0$$

is

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t).$$

This motion is called simple harmonic motion. The reason for this is that $x(t)$ can be rewritten as

$$x(t) = C \cos(\omega_0 t - \alpha) = C \cos(\omega_0 (t - \delta))$$

in terms of an amplitude $C > 0$ and a phase angle α (or in terms of a time delay δ).

To see why functions of the form

$$x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)$$

are equal (for appropriate choices of constants) to ones of the form

$$x(t) = C \cos(\omega_0 t - \alpha)$$

we use the very important the addition angle trigonometry identities, in this case the addition angle for *cosine*: Consider the possible equality of functions

$$A \cos(\omega_0 t) + B \sin(\omega_0 t) = C \cos(\omega_0 t - \alpha).$$

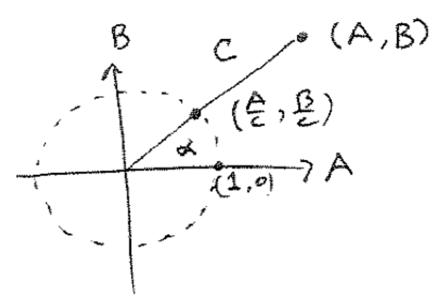
Exercise 5) Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to show that the two functions above are equal provided

$$A = C \cos \alpha$$
$$B = C \sin \alpha .$$

So if C, α are given, the formulas above determine A, B. Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha), \quad \frac{B}{C} = \sin(\alpha)$$

determine C, α. These correspondences are best remembered using a diagram in the $A - B$ plane:



It is important to understand the behavior of the functions

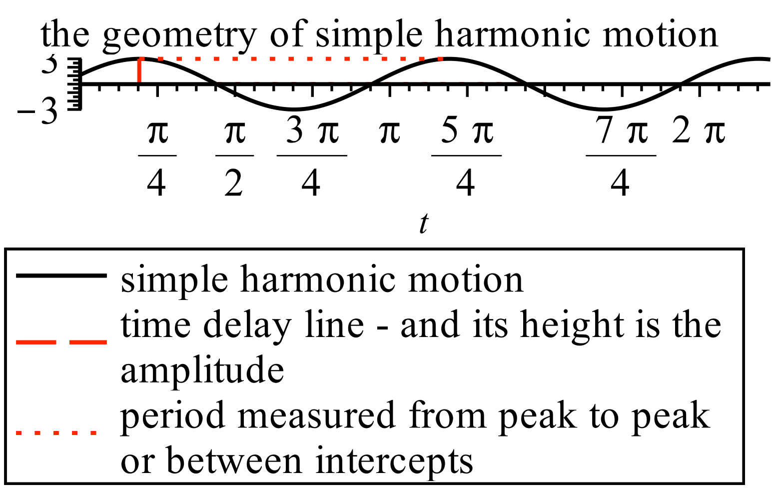
$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0(t - \delta))$$

and the standard terminology:

The amplitude C is the maximum absolute value of $x(t)$. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right in order to obtain the graph of $x(t)$. Other important data is

$$f = \text{frequency} = \frac{\omega_0}{2\pi} \quad \text{cycles/time}$$

$$T = \text{period} = \frac{2\pi}{\omega_0} = \text{time/cycle.}$$



(I made that plot above with these commands...and then added a title and a legend, from the plot options.)

```
> with(plots) :
> plot1 := plot(3·cos(2(t − .6)), t = 0 ..7, color = black) :
  plot2 := plot([.6, t, t = 0 ..3.], linestyle = dash) :
  plot3 := plot(3, t = .6 ..(.6) + Pi, linestyle = dot) :
  plot4 := plot(0.02, t = .6 + Pi/4 ...6 + 5·Pi/4, linestyle = dot) :
> display({plot1, plot2, plot3, plot4});
>
```

Exercise 6) A mass of 2 kg oscillates without damping on a spring with Hooke's constant $k = 18\ \frac{N}{m}$. It is initially stretched 1 m from equilibrium, and released with a velocity of $\frac{3}{2}\ \frac{m}{s}$.

6a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$x'' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

6b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the commands below.

```
> unassign(`x`);
> with(plots) :
  with(DEtools) :
> dsolve({x''(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
> plot(rhs(%), t = 0..5, color = green);
>
```

• Next, discuss the possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

Case 2: damping

$$m x'' + c x' + k x = 0$$
$$x'' + \frac{c}{m} x' + \frac{k}{m} x = 0$$

rewrite as

$$x'' + 2 p x' + \omega_0^2 x = 0.$$

$\left(p = \frac{c}{2m},\ \omega_0^2 = \frac{k}{m}\right)$. The characteristic polynomial is

$$r^2 + 2 p r + \omega_0^2 = 0$$

which has roots

$$r = -\frac{2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}.$$

2a) ($p^2 > \omega_0^2$, or $c^2 > 4mk$). overdamped. In this case we have two negative real roots

$$r_1 < r_2 < 0$$

and

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_2 t}\left(c_1 e^{(r_1 - r_2)t} + c_2\right).$$

- solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

2b) ($p^2 = \omega_0^2$, or $c^2 = 4mk$) critically damped. Double real root $r_1 = r_2 = -p = -\frac{c}{2m}$.

$$x(t) = e^{-pt}\left(c_1 + c_2 t\right).$$

- solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

2c) ($p^2 < \omega_0^2$, or $c^2 < 4mk$) underdamped. Complex roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i\,\omega_1$$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$.

$$x(t) = e^{-pt}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right) = e^{-pt} C\cos(\omega_1 t - \alpha_1).$$

- solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-pt}C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1.

Exercise 7) Classify by finding the roots of the characteristic polynomial. Then solve for $x(t)$:
7a)

$$x'' + 6\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

> *with*(*DEtools*) :

> $dsolve\left(\left\{x''(t) + 6\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

$$x(t) = \mathrm{e}^{-3\,t} + \frac{9}{2}\,\mathrm{e}^{-3\,t}\,t \quad \textbf{(2)}$$

>

7b)

$$x'' + 10\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

> $dsolve\left(\left\{x''(t) + 10\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

$$x(t) = -\frac{5}{16}\,\mathrm{e}^{-9\,t} + \frac{21}{16}\,\mathrm{e}^{-t} \quad \textbf{(3)}$$

>

7c)

$$x'' + 2\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

> $dsolve\left(\left\{x''(t) + 2\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

$$x(t) = \frac{5}{8}\,\sqrt{2}\,\mathrm{e}^{-t}\sin\left(2\,\sqrt{2}\,t\right) + \mathrm{e}^{-t}\cos\left(2\,\sqrt{2}\,t\right) \quad \textbf{(4)}$$

>

> *with*(*plots*) :

> $plot0 := plot\left(\cos(3\cdot t) + \frac{1}{2}\cdot\sin(3\cdot t), t = 0\,..4, color = red\right)$:

$$plot1a := plot\left(\exp(-3 \cdot t) \cdot \left(1 + \frac{9}{2} \cdot t\right), t = 0 ..4, color = green\right):$$

$$plot1b := plot\left(\frac{21}{16} \cdot \exp(-t) - \frac{5}{16} \cdot \exp(-9 \cdot t), t = 0 ..4, color = blue\right):$$

$$plot1c := plot\left(\frac{5}{8} \cdot \sqrt{2}\, e^{-t} \cdot \sin\left(2\sqrt{2} \cdot t\right) + e^{-t} \cdot \cos\left(2\sqrt{2} \cdot t\right), t = 0 ..4, color = black\right):$$

display({ *plot0*, *plot1a*, *plot1b*, *plot1c* }, *title* = \`*IVP with all damping possibilities*\`);