Math 2280-001
Week 3: Jan 23-27, sections 2.1-2.3

Mon Jan 23
2.1 Improved population models.

- Finish any remaining material from Friday.

- Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{people}{year}$) ;

$\beta(t) := \frac{B(t)}{P(t)}$, fertility rate ($\frac{people}{year}$ per *person*)

$\mathrm{D}(t)$, death rate (e.g. $\frac{people}{year}$);

$\delta(t) := \frac{\mathrm{D}(t)}{P(t)}$, mortality rate ($\frac{people}{year}$ per *person*)

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - \mathrm{D}(t)$$
$$P'(t) = \big(\beta(t) - \delta(t)\big)P(t) .$$

<u>Model 1:</u> constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \geq 0,\ \delta(t) \equiv \delta_0 \geq 0$, constants.

$$\Rightarrow P' = \left(\beta_0 - \delta_0\right)P = k\,P .$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

<u>Model 2:</u> population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$
$$\delta = \delta_0 + \delta_1 P$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = \left(\left(\beta_0 + \beta_1 P\right) - \left(\delta_0 + \delta_1 P\right)\right)P$$
$$= \left(\left(\beta_0 - \delta_0\right) + \left(\beta_1 - \delta_1\right)P\right)P .$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a < 0$, $\beta_0 - \delta_0 = b > 0$ one obtains the <u>logistic differential equation:</u>

$$P' = (b - a\,P)P$$
$$P' = -a\,P^2 + b\,P \text{, or equivalently}$$
$$P' = a\,P\left(\frac{b}{a} - P\right) = k\,P(M - P).$$

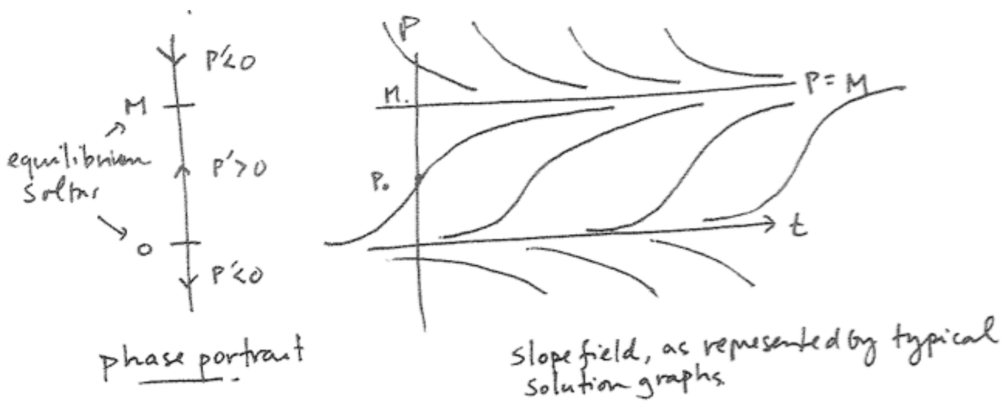
$k = a > 0$, $M = \frac{b}{a} > 0$. (One can consider other cases as well.)

Exercise 1: Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$:

$$P' = k\,P(M - P)$$

a) There are two constant ("equilibrium") solutions. What are they?

b) Evaluate the sign and magnitude of the slope function $f(P, t) = k\,P(M - P)$, in order to understand and be able to recreate the two diagrams below. One is a qualititative picture of the slope field, in the $t - P$ plane. The diagram to the left of it, called the phase diagram, is just a P number line with arrows indicating whether $P(t)$ is increasing or decreasing on the intervals between the constant solutions.



c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M. Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$?

Exercise 2: Solve the logistic DE IVP

$$P' = k\,P(M - P)$$
$$P(0) = P_0$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{\left(M - P_0\right)e^{-Mkt} + P_0}\ .$$

Solution (we will work this out step by step in class):

$$\frac{dP}{P(P - M)} = -k\,dt$$

By partial fractions,

$$\frac{1}{P(P - M)} = \frac{1}{M}\left(\frac{1}{P - M} - \frac{1}{P}\right).$$

Use this expansion and multiply both sides of the separated DE by M to obtain

$$\left(\frac{1}{P - M} - \frac{1}{P}\right)dP = -kM\,dt\ .$$

Integrate:

$$\ln|P - M| - \ln|P| = -Mkt + C_1$$
$$\ln\left|\frac{P - M}{P}\right| = -Mkt + C_1$$

exponentiate:

$$\left|\frac{P - M}{P}\right| = C_2\,e^{-Mkt}$$

Since the left-side is continuous

$$\frac{P - M}{P} = C\,e^{-Mkt} \quad (C = C_2 \text{ or } C = -C_2)$$

(At $t = 0$ we see that

$$\frac{P_0 - M}{P_0} = C\ .)$$

Now, solve for $P(t)$ by multiplying both sides of of the second to last equation by $P(t)$:

$$P - M = Ce^{-Mkt}P$$

Collect $P(t)$ terms on left, and add M to both sides:

$$P - Ce^{-Mkt}P = M$$
$$P\left(1 - Ce^{-Mkt}\right) = M$$
$$P = \frac{M}{1 - Ce^{-Mkt}}\ .$$

Plug in C and simplify:

$$P = \frac{M}{1 - \left(\frac{P_0 - M}{P_0}\right)e^{-Mkt}} = \frac{MP_0}{P_0 - \left(P_0 - M\right)e^{-Mkt}}$$

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}} .$$

Finally, because $\lim_{t \to \infty} e^{-Mkt} = 0$, we see that

$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

Note: If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0 , for $t > 0$.)

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Question: You have a couple of homework problems where you are asked to find solutions $x(t)$ to differential equations of the form

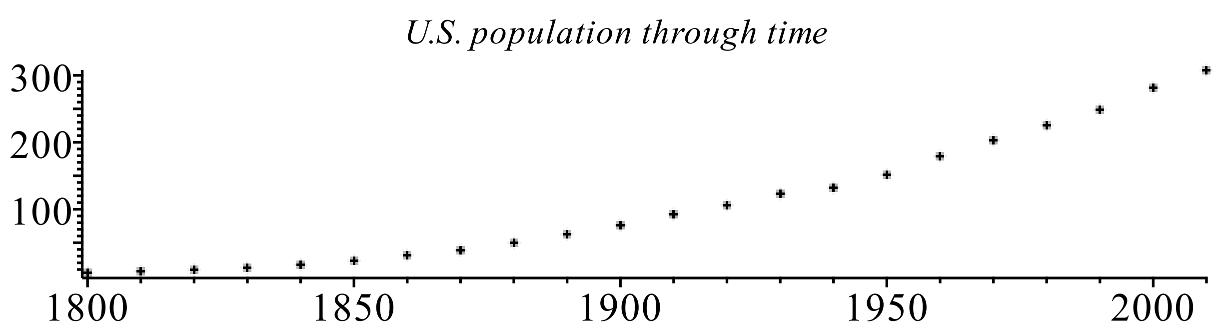
$$x'(t) = a(x - b) \cdot (x - c).$$

How would you proceed?

Application

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
      [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
      [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
      [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
      [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
      [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
       # I used shift-enter to enter more than one line of information
       # before executing the command.
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```




```
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0 e^{rt}$ we use the 1800 and 1900 data to get values for P_0 and r :

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
```

$$P0 := 5.308$$

$$0.026643 \quad (1)$$

```
> P1 := t→5.308· exp(.02664· t);#exponential model -eqtn (9) page 83
```

$$P1 := t \rightarrow 5.308\, e^{0.02664\, t} \quad (2)$$

```
>
```

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

```
> P2 := t→M·P0/(P0 + (M−P0)·exp(−M·k·t)); # logistic solution we worked out
```

$$P2 := t \rightarrow \frac{M\,P0}{P0 + (M - P0)\,e^{-M k t}} \quad (3)$$

```
> solve({P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12, k = 0.00016772\} \quad (4)$$

```
> M := 188.12;
  k := .16772e-3;
P2(t); #should be our logistic model function,
   #equation (11) page 84.
```

$$M := 188.12$$

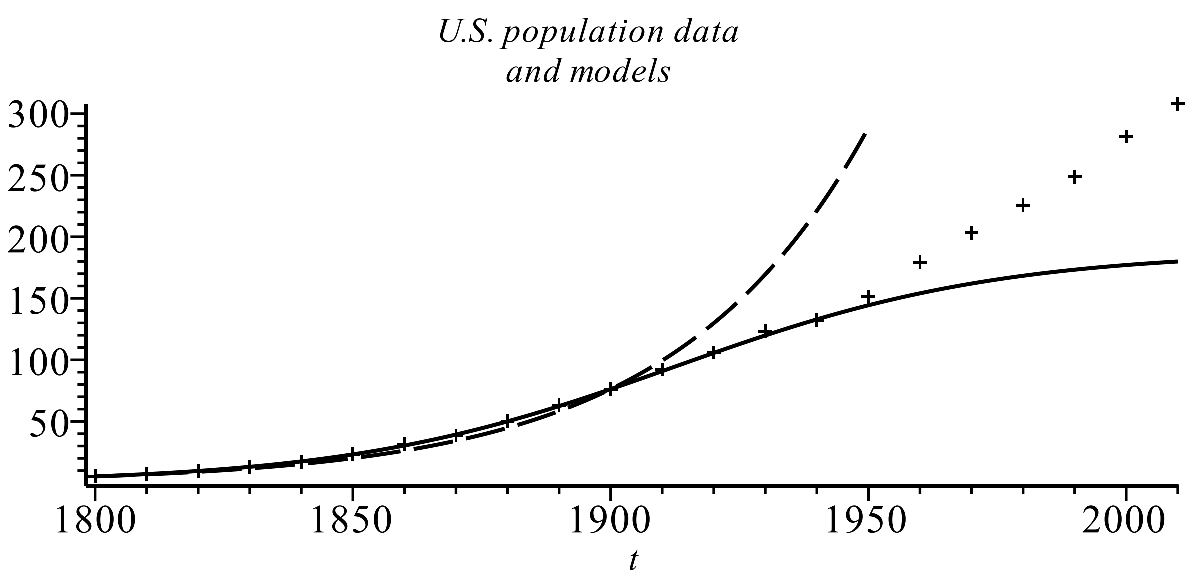
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\,e^{-0.031551\,t}} \quad (5)$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

```
> plot1 := plot(P1(t−1800), t = 1800..1950, color = black, linestyle = 3) :
   #this linestyle gives dashes for the exponential curve
plot2 := plot(P2(t−1800), t = 1800..2010, color = black) :
plot3 := pointplot(pops, symbol = cross) :
display({plot1, plot2, plot3}, title = `U.S. population data
and models`);
```




```
>
>
```

Math 2280-01
Wed Jan 28
2.2: Autonomous Differential Equations.

Recall, that a general first order DE for $x = x(t)$ is written in standard form as

$$x' = f(t, x) \,,$$

which is shorthand for $x'(t) = f(t, x(t))$.
Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x) \,.$$

Example: The logistic DE, $P' = k\,P(M - P)$ is an autonomous differential equation for $P(t)$.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 1: Find the equilibrium solutions of
1a) $x'(t) = 3\,x - x^2$

1b) $x'(t) = x^3 + 2\,x^2 + x$

1c) $x'(t) = \sin(x)$.

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

· c is a *stable* equilibrium solution if solutions with initial values close enough to c stay close to c. There is a precise way to say this, but it requires quantifiers: For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \varepsilon$ for all $t > 0$.

· c is an *unstable* equilibrium if it is not stable.

· c is an *asymptotically stable* equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c, then $\lim_{t \to \infty} x(t) = c$, i.e. there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then $\lim_{t \to \infty} x(t) = c$. (Notice that this means the horizontal line $x = c$ will be an *asymptote* to the solution graphs $x = x(t)$ in these cases.)

Exercise 2: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1. For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

2a) $x'(t) = 3x - x^2$

2b) $x'(t) = x^3 + 2x^2 + x$

2c) $x'(t) = \sin(x)$.

Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e. $x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of f, i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can is completely determined by the local phase diagrams:

$sign(f)$: $- - - - 0 + + +$ $\Rightarrow$ $\leftarrow\leftarrow\leftarrow c \rightarrow\rightarrow\rightarrow$ $\Rightarrow$ c is unstable

$sign(f)$: $+ + + 0 - - - -$ $\Rightarrow$ $\rightarrow\rightarrow\rightarrow c \leftarrow\leftarrow\leftarrow$ $\Rightarrow$ c is asymptotically stable

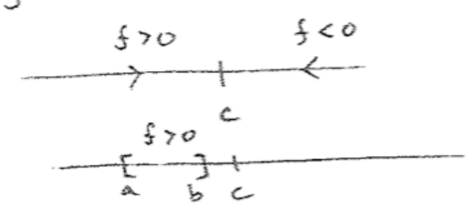
$sign(f)$: $+ + + 0 + + + +$ $\Rightarrow$ $\rightarrow\rightarrow\rightarrow c \rightarrow\rightarrow\rightarrow$ $\Rightarrow$ c is unstable (half stable)

$sign(f)$: $- - - - 0 - - - -$ $\Rightarrow$ $\leftarrow\leftarrow\leftarrow c \leftarrow\leftarrow\leftarrow$ $\Rightarrow$ c is unstable (half stable)

You can actually prove this Theorem with calculus!! (want to try?)

Here's why!

e.g. consider the second case



f cont; $f > 0$ on subinterval $[a,b]$

$\Rightarrow f \geq \delta > 0$ on $[a,b]$

(extreme value thm from calculus, f attains its minimum)

$\Rightarrow x'(t) \geq \delta$ as long as $x(t) \in [a,b]$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$ ■

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(t - c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

Further application: Doomsday-extinction. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

Logistic: $P'(t) = -a\,P^2 + b\,P$

Doomsday-extinction: $Q'(t) = a\,Q^2 - b\,Q$

For example, suppose that the chances of procreation are proportional to population density (think alligators or crickets), i.e. the fertility rate $\beta = a\,Q(t)$, where $Q(t)$ is the population at time t. Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are $a\,Q^2$ and $-b\,Q$ .... which yields the DE above. In this case factor the right side:

$$Q'(t) = a\,Q\left(Q - \frac{b}{a}\right) = k\,Q(Q - M).$$

Exercise 4a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilbrium solutions.

Exercise 4b) If $P(t)$ solves the logistic differential equation

$$P'(t) = k\,P(M - P)$$

show that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation

$$Q'(t) = k\,Q(Q - M)\ .$$

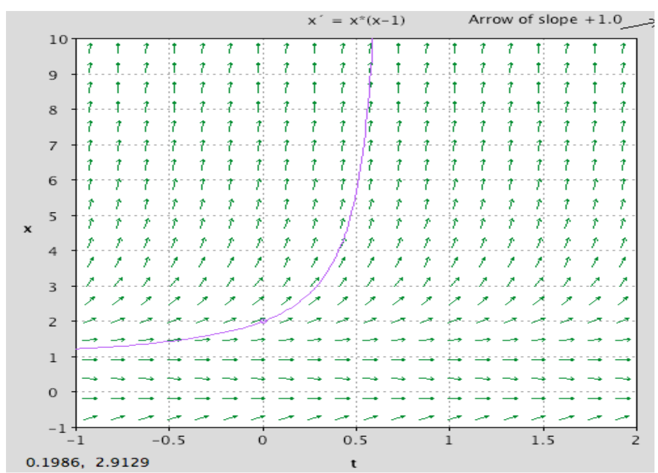
Use this to recover a formula for solutions to doomsday-extinction IVPs. What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models? Recall, the solution to the logistic IVP is

$$P(t) = \frac{MP_0}{\left(M - P_0\right)e^{-Mkt} + P_0}\ .$$

Exercise 5: Use your formula from the previous exercise or work the separable DE from scratch, to transcribe the solution to the doomsday-extinction IVP

$$x'(t) = x(x - 1)$$
$$x(0) = 2\ .$$

Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)



Math 2280-001
Fri Jan 30
2.2 Autonomous differential equations, with applications; 2.3 improved velocity models

- Recall that on Wednesday we discussed the following important concepts:
  - * Autonomous first order DE
  - * equilibrium solutions for autonomous DE's
  - * stability at equilibrium points.

Further application: (related to parts of a "yeast bioreactor" homework problem for next week) harvesting a logistic population...text p.89-91 (or, why do fisheries sometimes seem to die out "suddenly"?)
Consider the DE

$$P'(t) = a\,P - b\,P^2 - h\,.$$

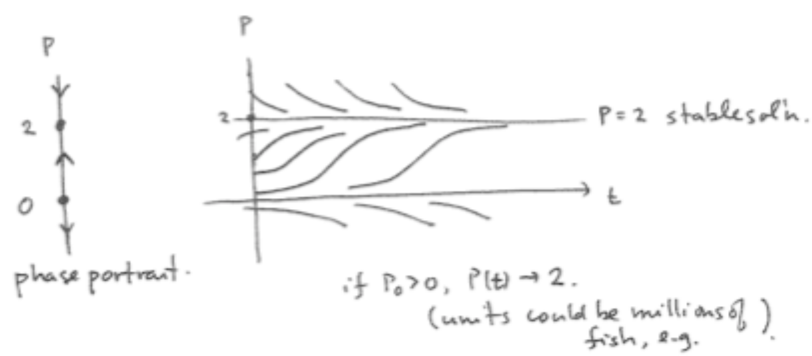
Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be $h\,P$ instead of P.

For computational ease we will assume $a = 2$, $b = 1$. (One could actually change units of population and time to reduce to this case.)

for computational simplicity take $a=2$, $b=1$

Case 0 no harvesting

$$P'(t) = 2P - P^2 = P(2-P)$$



with harvesting:

$$\begin{aligned} P'(t) &= 2P - P^2 - h \\ &= -(P^2 - 2P + h) \\ &= -(P-P_1)(P-P_2) \end{aligned}$$

$$P_1, P_2 = \frac{2 \pm \sqrt{4-4h}}{2} = 1 \pm \sqrt{1-h}$$

Case 1: subcritical harvesting
$0 < h < 1$

Case 2: Critical harvesting

$h = 1$

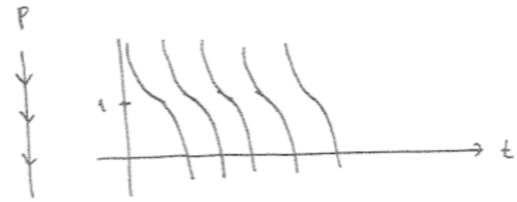
$P'(t) = -(P-1)^2$

Case 3 Over harvesting

$h > 1$

complex roots.

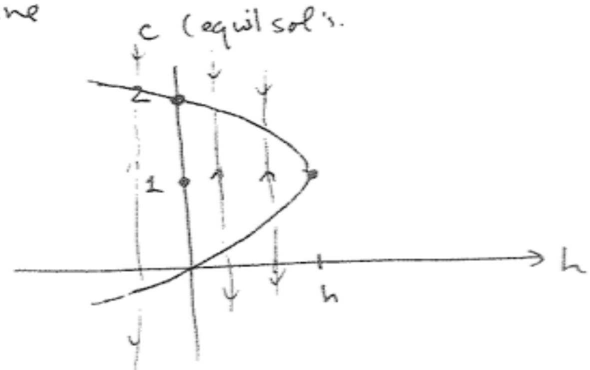
$$P'(t) = -(P^2 - 2P + h) = -[(P-1)^2 + (h-1)] < 0.$$



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \rightarrow 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h . The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h- axis should be labeled $h = 1$, not h . What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1 - h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2 - c)$.

"bifurcation diagram" of equilibrium soltns in the h-c plane



2.3 Improved velocity models: velocity-dependent drag forces

For particle motion along a line, with

position $x(t)$ (or $y(t)$),
velocity $x'(t) = v(t)$, and
acceleration $x''(t) = v'(t) = a(t)$

We have Newton's 2^{nd} law

$$m\, v'(t) = F$$

where F is the net force.

- We're very familiar with constant force $F = m\,\alpha$, where α is a constant:

$$v'(t) = \alpha$$
$$v(t) = \alpha\, t + v_0$$
$$x(t) = \frac{1}{2}\alpha\, t^2 + v_0\, t + x_0 .$$

Examples we've seen a lot of:

- $\alpha = -g$ near the surface of the earth, if up is the positive direction, or $\alpha = g$ if down is the positive direction.
- boats or cars or "particles" subject to constant acceleration or deceleration.

New today !!! Combine a constant force with a velocity-dependent drag force, at the same time. The text calls this a "resistance" force:

$$m\, v'(t) = m\,\alpha + F_R$$

Empirically/mathematically the resistance forces F_R depend on velocity, in such a way that their magnitude is

$$|F_R| \approx k\,|v|^p \ , 1 \le p \le 2 .$$

- $p = 1$ (linear model, drag proportional to velocity):

$$m\, v'(t) = m\,\alpha - k\, v$$

This linear model makes sense for "slow" velocities, as a linearization of the frictional force function, assuming that the force function is differentiable with respect to velocity...recall Taylor series for how the velocity resistance force might depend on velocity:

$$F_R(v) = F_R(0) + F_R'(0)\, v + \frac{1}{2!}F_R''(0)v^2 + ...$$

$F_R(0) = 0$ and for small enough v the higher order terms might be negligable compared to the linear term, so

$$F_R(v) \approx F_R'(0)\, v \approx -k\, v .$$

We write $-k\, v$ with $k > 0$, since the frictional force opposes the direction of motion, so sign opposite of the velocity's.

http://en.wikipedia.org/wiki/Drag_(physics)#Very_low_Reynolds_numbers:_Stokes.27_drag

Exercise 1a: Rewrite the linear drag model as

$$v'(t) = \alpha - \rho v$$

where the $\rho = \dfrac{k}{m}$. Construct the phase diagram for v . Notice that $v(t)$ has exactly one constant (equilibrium) solution, and find it. Its value is called the *terminal velocity*. Explain why *terminal velocity* is an appropriate term of art, based on your phase diagram.

1b) Solve the IVP

$$v'(t) = \alpha - \rho v$$
$$v(0) = v_0$$

and verify your phase diagram analysis. (This is, once again, our friend the first order constant coefficient linear differential equation.)

1c) integrate the velocity function above to find a formula for the position function $y(t)$.

- $p = 2$, for the power in the resistance force. This can be an appropriate model for velocities which are not "near" zero....described in terms of "Reynolds number". Accounting for the fact that the resistance opposes direction of motion we get

$$m\, v'(t) = m\,\alpha - k\,v^2 \quad \text{if } v > 0$$
$$m\, v'(t) = m\,\alpha + k\,v^2 \quad \text{if } v < 0.$$

http://en.wikipedia.org/wiki/Drag_(physics)#Drag_at_high_velocity

Exercise 2) Once again letting $\rho = \frac{k}{m}$ we can rewrite the DE's as

$$v'(t) = \alpha - \rho\,v^2 \quad \text{if } v > 0$$
$$v'(t) = \alpha + \rho\,v^2 \quad \text{if } v < 0.$$

2a) Consider the case in which $\alpha = -g$, so we are considering vertical motion, with up being the positive direction. Draw the phase diagrams. Note that each diagram contains a half line of v-values. Make conclusions about velocity behavior in case $v_0 > 0$ and $v_0 \le 0$. Is there a terminal velocity?

2b) Set up the two separable differential equation IVPs for the cases above, so that you will be able to complete finding the solution formulas (in your homework)....Of course, once you find the velocity function you'll still need to integrate that, if you want to find the position function!

Application: We consider the bow and deadbolt example from the text, page 102-104. It's shot vertically into the air (watch out below!), with an initial velocity of 49 $\frac{m}{s}$. In the no-drag case, this could just be the vertical component of a deadbolt shot at an angle. With drag, one would need to study a more complicated system of DE's for the horizontal and vertical motions, if you didn't shoot the bolt straight up.

Exercise 3: First consider the case of no drag, so the governing equations are

$$v'(t) = -g \approx -9.8\ \frac{m}{s^2}$$
$$v(t) = -g\,t + v_0 = -g\,t + 5\,g$$
$$x(t) = -\frac{1}{2}g\,t^2 + v_0\,t + x_0 = -\frac{1}{2}g\,t^2 + 5\,g\,t\,.$$

Find when $v = 0$ and deduce how long the object rises, how long it falls, and its maximum height.

Maple check:

```
> restart:
  Digits := 5:
```

```
> g := 9.8;
  v0 := 49.0;
  v1 := t→-g·t + v0;
  y1 := t→-1/2·g·t^2 + v0·t;
```

$$g := 9.8$$
$$v0 := 49.0$$
$$v1 := t \rightarrow -g\,t + v0$$
$$y1 := t \rightarrow -\frac{1}{2}\,g\,t^2 + v0\,t \qquad \textbf{(6)}$$

Exercise 4: Now consider the linear drag model for the same deadbolt, with the same initial velocity of $5\,g = 49\ \frac{m}{s}$. We'll assume that our deadbolt has a measured terminal velocity of $v_\tau = -245\ \frac{m}{s} = -25\,g$, so $|v_\tau| = 25\,g = \frac{g}{\rho} \Rightarrow \rho = .04$ (convenient). So, from our earlier work:

$$v = v_\tau + (v_0 - v_\tau)\,\mathrm{e}^{-\rho t}$$

$$y = y_0 + t\,v_\tau + \frac{(v_0 - v_\tau)\left(1 - \mathrm{e}^{-\rho t}\right)}{\rho}$$

So,

v

$$v = -\frac{g}{\rho} + \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} = -245 + 294\,e^{-.04\,t}.$$

$$y = 0 - 245\,t + \frac{294}{.04}\left(1 - \mathrm{e}^{-.04\,t}\right).$$

When does the object reach its maximum height, what is this height, and how long does the object fall? Compare to the no-drag case with the same initial velocity, in Exercise 3.

Maple check, and then work:

```
>
> with(DEtools):
> g := 9.8; ρ := .04; v0 := 49;
```

$$g := 9.8$$
$$\rho := 0.04$$
$$v0 := 49 \qquad \textbf{(7)}$$

```
> dsolve({v'(t) = -g - ρ·v(t), v(0) = v0}, v(t));
```

$$v(t) = -245 + 294\,\mathrm{e}^{-\frac{1}{25}t} \qquad \textbf{(8)}$$

```
> v2 := t → -245.0 + 294 e^(-1/25 t);
```

$$v2 := t \rightarrow -245.0 + 294\,\mathrm{e}^{-\frac{1}{25}t} \qquad \textbf{(9)}$$

```
> solve(v2(t) = 0, t);
```

$$4.5580 \qquad \textbf{(10)}$$

```
> y2 := t → y0 + ∫_0^t -g/ρ + e^(-ρ s) (v0 + g/ρ) ds;
```

$$y2 := t \rightarrow y0 + \int_0^t \left(-\frac{g}{\rho} + \mathrm{e}^{-\rho s}\left(v0 + \frac{g}{\rho}\right)\right) \mathrm{d}s \qquad \textbf{(11)}$$

```
> v2(t);
  294/.04;
  y0 := 0;
  y2(t);
```

$$-245.0 + 294\, e^{-\frac{1}{25} t}$$

$$7350.0$$

$$y0 := 0$$

$$7350. - 245.\, t - 7350.\, e^{-0.040000\, t} \qquad \textbf{(12)}$$

```
> solve(v2(t) = 0, t);
  solve(y2(t) = 0, t);
  y2(4.558);
```

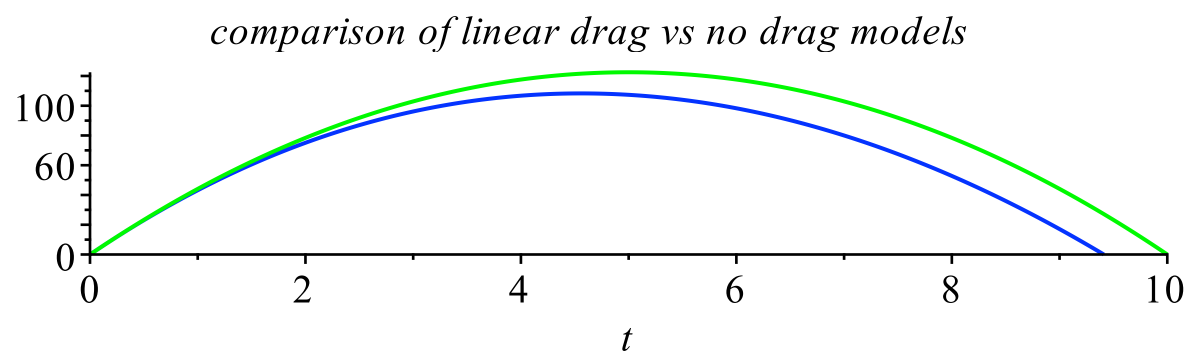
$$4.5580$$

$$9.4110,\ 0.$$

$$108.28 \qquad \textbf{(13)}$$

picture:

```
> with(plots) :
  plot1 := plot(y1(t), t = 0 ..10, color = green) :
  plot2 := plot(y2(t), t = 0 ..9.4110, color = blue) :
  display({plot1, plot2}, title = `comparison of linear drag vs no drag models`);
```




```
>
```