Math 2280-001
Week 14, April 17-21 and Week 15, April 24:
9.1-9.4 Fourier series and forced oscillations revisited; 6.1-6.4 nonlinear autonomous systems.
Mon Apr 17

<u>9.1-9.3</u> Fourier Series. (On Wednesday we'll revisit forced oscillations, section 9.4, and explain the results
we obtained playing "the resonance game" with convolution integrals last Wednesday.)

•   Finish Friday's notes if necessary. The key points to recall from Friday are that Fourier series are a
way of expressing piecewise continuous functions f on the interval $[-\pi, \pi]$ (or equivalently, 2π-periodic
functions f), as infinite sum of trigonometric functions. If f has Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,t)$$

Then the partial sums

$$f_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

with

$$a_0 := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt \qquad \left(\frac{a_0}{2} = <f, \frac{1}{\sqrt{2}}> \frac{1}{\sqrt{2}} \right)$$

$$a_n := \langle f, \cos(n\,t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(n\,t)\, dt, \; n \in \mathbb{N}$$

$$b_n := \langle f, \sin(n\,t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(n\,t)\, dt, \; n \in \mathbb{N}$$

are the projections of f onto

$$V_N = span\left\{ \frac{1}{\sqrt{2}}, \cos(t), \cos(2\,t), ..., \cos(N\,t), \sin(t), \sin(2\,t), ... \sin(N\,t) \right\}.$$

These partial sums converge to f as $N \to \infty$ in the ways described in Friday's notes.

•   After finishing with Friday's notes, continue with today's...

Exercise 1 If a function ("vector") is already in a subspace, then projection onto that subspace leaves the function fixed. Use that fact to very quickly compute the 2π-periodic Fourier series for

a) $f(t) = \sin(5\,t) - 8\cos(10\,t)$

b) $g(t) = \cos^2(3\,t)$

Fourier series for 2 L-periodic functions:

Theorem: Consider the vector space of piecewise continuous, 2 L-periodic functions. Then the inner product

$$\langle g, h \rangle := \frac{1}{L} \int_{-L}^{L} g(u)h(u) \, du$$

makes

$$\left\{ \frac{1}{\sqrt{2}}, \cos\left(\frac{\pi}{L} u \right), \cos\left(\frac{2\pi}{L} u \right), \dots \cos\left(\frac{k\pi}{L} u \right), \dots \sin\left(\frac{\pi}{L} u \right), \sin\left(\frac{2\pi}{L} u \right), \dots \sin\left(\frac{k\pi}{L} u \right) \dots \right\}$$

into an orthonormal collection of functions.

proof: The substitution $\frac{\pi}{L} u = t$, equivalently $u = \frac{L}{\pi} t$ converts between 2 L-periodic functions $g(u), h(u)$ and 2 π-periodic functions $g\left(\frac{L}{\pi} t \right), h\left(\frac{L}{\pi} t \right)$. Also (verify this!!!)

$$\frac{1}{L} \int_{-L}^{L} g(u)h(u) \, du = \frac{1}{\pi} \int_{-\pi}^{\pi} g\left(\frac{L}{\pi} t \right) h\left(\frac{L}{\pi} t \right) dt \, .$$

$\square$

Thus the Fourier series for a 2 L- periodic function f is defined by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \frac{\pi}{L} u \right) + \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} u \right)$$
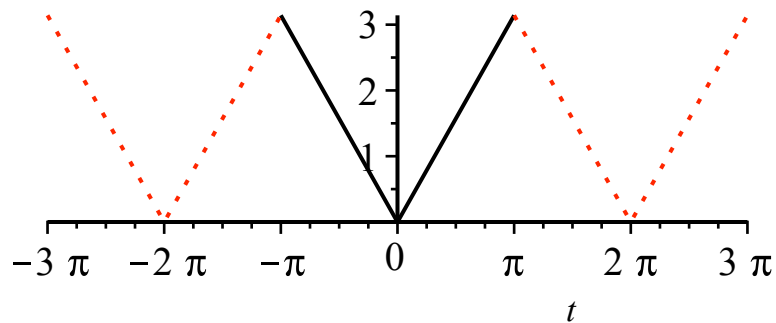
with

$$a_0 = \frac{1}{L} \int_{-L}^{L} g(u) \, du$$

$$a_n := \left\langle f, \cos\left(n \frac{\pi}{L} u \right) \right\rangle = \frac{1}{L} \int_{-L}^{L} f(u) \cos\left(n \frac{\pi}{L} u \right) du, \ n \in \mathbb{N}$$

$$b_n := \left\langle f, \sin\left(n \frac{\pi}{L} u \right) \right\rangle = \frac{1}{L} \int_{-L}^{L} f(u) \sin\left(\frac{\pi}{L} u \right) du, \ n \in \mathbb{N}$$

(and then we usually use the dummy variable t rather than u). As a result, Fourier series for 2 L-periodic functions along with convergence theorems, are "equivalent" to ones for 2 π-periodic ones, via this *isometry* of the two vector spaces.
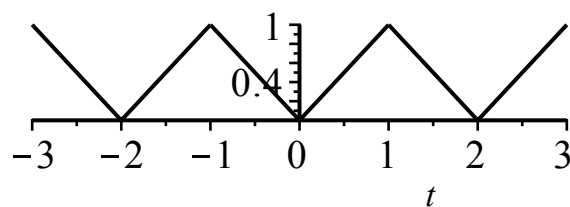
Exercise 2)

<u>a)</u> Use the Fourier series for *tent(t)*:

$$tent \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n\,odd} \frac{1}{n^2} \cos(n\,t)$$

to deduce the Fourier series for the related function $f(u)$ with period 2 that has graph

solution: $f \sim \dfrac{1}{2} - \dfrac{4}{\pi^2} \sum_{n\,odd} \dfrac{1}{n^2} \cos(n\,\pi\,u)$

We talked about differentiating Fourier series term by term in Friday's notes. There is also:

Theorem If f is piece-wise continuous, $2\,L$-periodic, with Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}t\right)$$

then the antiderivative may be computed by term by term antidifferentiation, and the corresponding series will converge for each t:

$$\int_0^t f(s)\,\mathrm{d}s = \frac{a_0}{2}t + \sum_{n=1}^{\infty} a_n\left(\frac{L}{n\,\pi}\right)\sin\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n\left(\frac{L}{n\,\pi}\right)\left(-\cos\left(n\frac{\pi}{L}t\right) + 1\right).$$

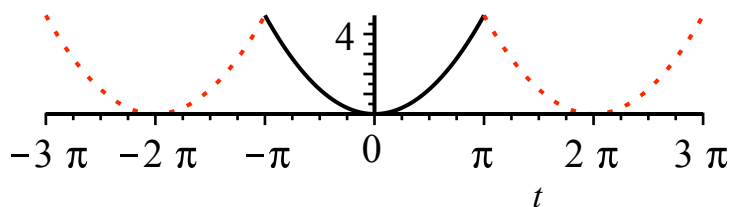Exercise 3) (This is the first part of your homework exercise 9.3.19)
Start with

$$t = saw(t) \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\,t)$$

and integrate to get the $2\,\pi$- periodic function that on $[-\pi, \pi]$ is given by

$$\frac{t^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos(n\,t).$$

Hint: The value of the constant term is easiest to compute as $\dfrac{a_0}{2}$. If you compare to the definite integral formula in the Theorem you will reproduce one of the "magic" series.

In your homework you will antidifferentiate twice more to get a formula for the periodic extension of

$g(t) = \dfrac{t^4}{24}$ and some more magic formulas.

Math 2280-001
Wed Apr 19

· Finish Monday's notes first. Then ...

9.4 Forced oscillation problems via Fourier Series. Today we will revisit the forced oscillation problems of last Friday, where we predicted whether or not resonance would occur, and then tested our predictions with the convolution solutions. Using Fourier series expansions for the forcing function one can say precisely whether or not there will be resonance. We will be studying the differential equations

$$x''(t) + c\,x'(t) + \omega_0^2 x(t) = f(t)$$

for various forcing functions $f(t)$. (We have divided the original mass-spring DE by the mass m and relabeled the damping coefficient and forcing functions.) For most of the lecture we consider undamped configurations, $c = 0$.

Warm-up Exercise 1) (This was the final exercise last Wednesday, when we were using convolution integrals to study forced oscillation problems.)
Use superposition to find particular solutions, and discuss whether or not resonance will occur in the following two forced oscillation problems. Notice that the period of the forcing function is 6π in a (not the natural period). In b the period of the forcing function is 2π. And yet, the resonance occurs in a, and not in b.

a)

$$x''(t) + x(t) = \cos(t) + \sin\left(\frac{t}{3}\right).$$

b)

$$x''(t) + x(t) = \cos(2\,t) - 3\,\sin(3\,t) + \cos(6\,t).$$

Hint: There's a table of particular solutions at the end of today's notes.

Exercise 1 is indicative of how we can understand resonance phenomena for forced oscillation problems with general periodic forcing functions f:

$$x''(t) + \omega_0^2 \, x(t) = f(t),$$

where $f(t)$ has period $P = 2L$. Compute the Fourier series for f:

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi}{L}t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}t\right)$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\mathrm{d}t \qquad \left(\text{so } \frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(t)\mathrm{d}t \text{ is the average value of } f\right)$$

$$a_n := \left\langle f, \cos\left(n\frac{\pi}{L}t\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n\frac{\pi}{L}t\right)\mathrm{d}t, \ n \in \mathbb{N}$$

$$b_n := \left\langle f, \sin\left(n\frac{\pi}{L}t\right)\right\rangle = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(n\frac{\pi}{L}t\right)\mathrm{d}u, \ n \in \mathbb{N}$$

As long as no (non-zero) term in the Fourier series has an angular frequency of ω_0, there will be no resonance. In fact, in this case the infinite sum of (undetermined coefficients) particular solutions will converge to a bounded particular solution. For sure there will NOT be resonance if it's true for all $n \in \mathbb{N}$ that

$$\omega_n := n\frac{\pi}{L} \neq \omega_0$$

but even if some ω_n does equal ω_0 there won't be resonance unless either a_n or b_n is nonzero.

Conversely, if the Fourier series of f does contain $\cos\left(\omega_0 t\right)$ or $\sin\left(\omega_0 t\right)$ terms, those terms will cause resonance.

Recall the first "resonance game" example from last Friday:
$$x''(t) + x(t) = square(t)$$
with
$$square(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

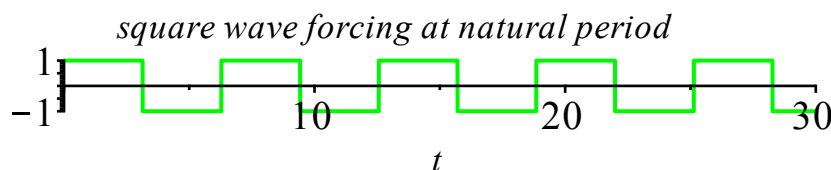and 2π-periodic. This forcing function appeared to cause resonance:
Here's a formula for $square(t)$ valid for $0 \le t \le 11\pi$, and last Friday's results:

> $with(plots)$:

> $square := t \to 1 + 2 \cdot \left(\displaystyle\sum_{n=1}^{10} (-1)^n \cdot \text{Heaviside}(t - n \cdot \text{Pi}) \right)$:

$plot1a := plot(square(t), t = 0..30, color = green)$:
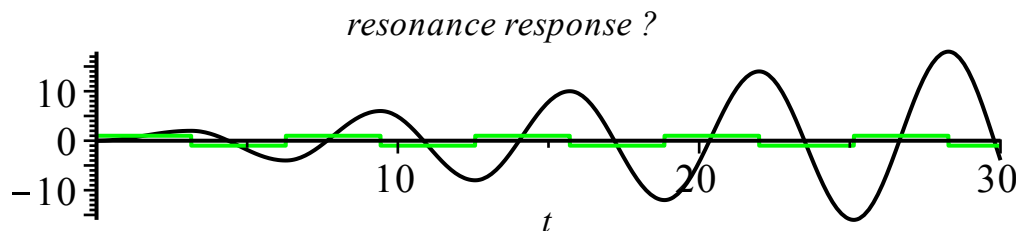$display(plot1a, title = $ `square wave forcing at natural period`$)$;

square wave forcing at natural period

Convolution solution formula and graph:

> $x1 := t \to \displaystyle\int_0^t \sin(\tau) \cdot square(t - \tau) \, d\tau$:

$plot1b := plot(x1(t), t = 0..30, color = black)$:
$display(\{plot1a, plot1b\}, title = $ `resonance response ?`$)$;

resonance response ?

Exercise 2 Use the Fourier series for $square(t)$ that we've found before
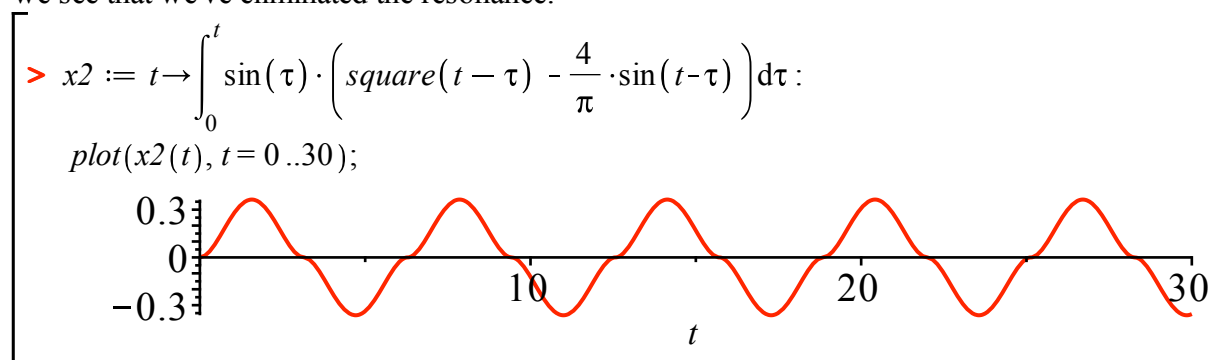
$$square(t) = \frac{4}{\pi} \sum_{n \, odd} \frac{1}{n} \sin(n \, t)$$

and infinite superposition to find a particular solution to
$$x''(t) + x(t) = square(t)$$
that explains why resonance occurs. Make use of the undetermined coefficients particular solution formulas at the end of today's notes.

If we remove the $\sin(t)$ term from the square wave forcing function, and re-use the convolution formula, we see that we've eliminated the resonance:
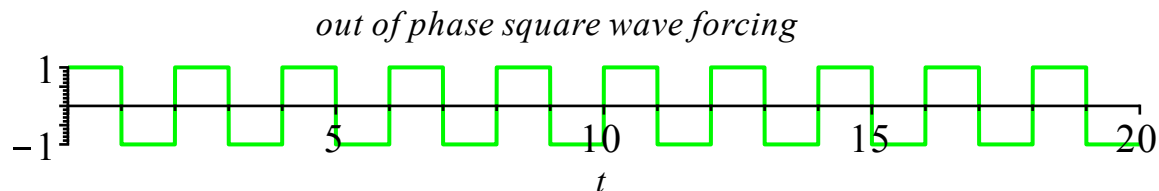
> $x2 := t \rightarrow \displaystyle\int_{0}^{t} \sin(\tau) \cdot \left(square(t - \tau) - \dfrac{4}{\pi} \cdot \sin(t - \tau) \right) d\tau :$

$plot(x2(t), t = 0 .. 30);$

Exercise 2) Understand Example 3 from last Friday, using Fourier series:
$$x''(t) + x(t) = f_3(t)$$

Example 3) Forcing not at the natural period, e.g. with a square wave having period $T = 2$.

> $f3 := t \to 1 + 2 \cdot \sum_{n=1}^{20} (-1)^n \cdot \text{Heaviside}(t - n)$:

$plot3a := plot(f3(t), t = 0 ..20, color = green)$:
$display(plot3a, title = \text{`out of phase square wave forcing`})$;

out of phase square wave forcing

This forcing function did not cause resonance:

> $x3 := t \to \int_0^t \sin(\tau) \cdot f3(t - \tau) \, d\tau$:

$plot3b := plot(x3(t), t = 0 ..20, color = black)$:
$display(\{plot3a, plot3b\}, title = \text{`resonance response ?`})$;

resonance response ?
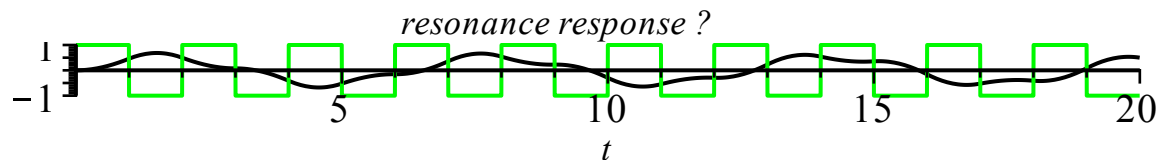
Hint: By rescaling we can express $f_3(t) = square(\pi t) = \dfrac{4}{\pi} \sum_{n \text{ odd}} \dfrac{1}{n} \sin(n \pi t)$.

Brute force tech check of Fourier coefficients in previous example:

```
> f := t → -1 + 2·Heaviside(t);
  plot(f(t), t = -1 .. 1);
  L := 1;
```

$$f := t \to -1 + 2\,\text{Heaviside}(t)$$

$$L := 1 \tag{1}$$
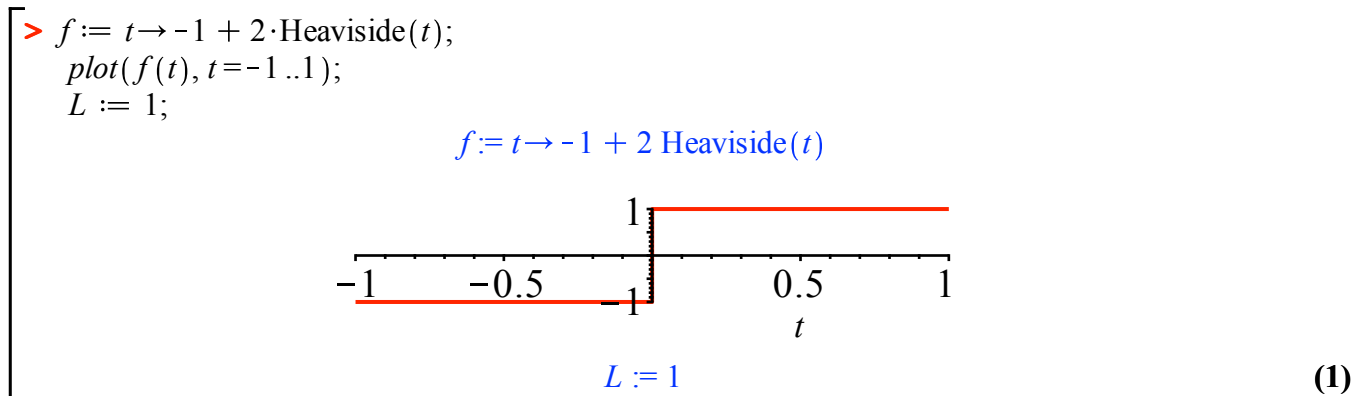
```
> a0 := 1/L · ∫_{-L}^{L} f(t) dt;

  assume(n, integer);  # this will let Maple attempt to evaluate the integrals

  a := n → 1/L · ∫_{-L}^{L} f(t)·cos( n·π/L t ) dt :

  b := n → 1/L · ∫_{-L}^{L} f(t)·sin( n·π/L t ) dt :

  a(n);
  b(n);
```

$$a0 := 0$$
$$0$$
$$\frac{-(-1)^{n\sim} + 2}{n\sim\,\pi} - \frac{(-1)^{n\sim}}{n\sim\,\pi} \tag{2}$$

```
>
>
```

Practical resonance example:

<u>Exercise 3</u> The steady periodic solution to the differential equation
$$x''(t) + .2 \cdot x'(t) + 1\,x(t) = square(t)$$
exhibits practical resonance. Explain this with Fourier series. Hint: Use

$$square(t) = \frac{4}{\pi} \sum_{n\,odd} \frac{1}{n} \sin(n\,t)$$

the table of particular solutions at the end of today's notes.

Particular solutions from Chapter 3 or Laplace transform table:

$$x''(t) + \omega_0^2 x(t) = A \sin(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2} \sin(\omega t) \quad \text{when } \omega \neq \omega_0$$

$$x_P(t) = -\frac{t}{2\omega_0} A \cos(\omega_0 t) \quad \text{when } \omega = \omega_0$$

....

$$x''(t) + \omega_0^2 x(t) = A \cos(\omega t)$$

$$x_P(t) = \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t) \quad \text{when } \omega \neq \omega_0$$

$$x_P(t) = \frac{t}{2\omega_0} A \sin(\omega_0 t) \quad \text{when } \omega = \omega_0$$

....

$$x'' + c x' + \omega_0^2 x = A \cos(\omega t) \quad c > 0$$

$$x_P(t) = x_{sp}(t) = C \cos(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \quad .$$

$$\cos(\alpha) = \frac{\omega_0^2 - \omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \quad .$$

....

$$x'' + c x' + \omega_0^2 x = A \sin(\omega t) \quad c > 0$$

$$x_P(t) = x_{sp}(t) = C \sin(\omega t - \alpha)$$

with

$$C = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}} \quad .$$

$$\cos(\alpha) = \frac{\omega^2 - \omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}$$

Fri Apr 21 and Mon Apr 24

6.1-6.4 Nonlinear systems of autonomous first order differential equations and applications.

Introduction: In Chapter 2 we talked about equilibrium solutions to autonomous differential equations, i.e. constant solutions. Constant solutions are important because in real world dynamics the dynamics of a system are often only varying slightly from a constant values, especially if the constant configuration is stable. And whenever the situation is nearly in equilibrium, one can understand the dynamical system behavior by linearization. And, it turns out that the best way to understand equilibrium solutions is often to convert autonomous differential equations or systems to first order systems.

Example 1) The rigid rod pendulum.

We've already considered a special case of this configuration, when the angle θ from vertical is near zero. Now assume that the pendulum is free to rotate through any angle $\theta \in \mathbb{R}$.
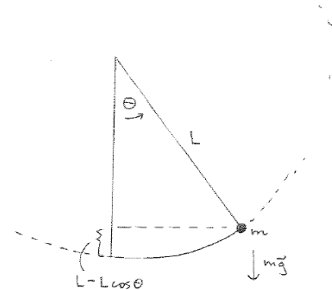
Earlier in the course we used conservation of energy to derive the dynamics for this (now) swinging, or possibly rotating, pendulum. There were no assumptions about the values of θ in that derivation of the non-linear DE (it was only when we linearized that we assumed θ was near zero). We began with the total energy

$$TE = KE + PE = \frac{1}{2}mv^2 + mgh$$

$$= \frac{1}{2}m(L\,\theta'(t))^2 + m\,g\,L\left(1 - \cos(\theta(t))\right)$$

And set $TE'(t) \equiv 0$ to arrive at the differential equation

$$\theta''(t) + \frac{g}{L}\sin\left(\theta(t)\right) = 0 \ .$$

We see that the constant solutions $\theta(t) = \theta_*$ must satisfy $\sin\left(\theta_*\right) = 0$, i.e. $\theta_* = n\,\pi, \pi \in \mathbb{R}$. In other words, the mass can be at rest at the lowest possible point (if θ is an even multiple of π), but also at the highest possible point (if θ is any odd multiple of π). We expect the lowest point configuration to be a "stable" constant solution, and the other one to be "unstable".

We will study these stability questions systematically using the equivalent first order system for

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix}$$

when $\theta(t)$ represents solutions the pendulum problem. You can quickly check that this is the system

$$x'(t) = y$$
$$y'(t) = -\frac{g}{L}\sin(x).$$

Notice that constant solutions of this system, $x' \equiv 0$, $y' \equiv 0$, equivalently

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_* \\ y_* \end{bmatrix} \text{ equals constant}$$

must satisfy $y_* = 0$, $\sin(x_*) = 0$, In other words, $x = n\,\pi$, $y = 0$ are the equilibrium solutions. These correspond to the constant solutions of the second order pendulum differential equation, $\theta = n\,\pi$, $\theta' = 0$.

Here's a phase portrait for the first order pendulum system, with $\frac{g}{L} = 1$, see below.

a) Locate the equilibrium points on the picture.

b) Interpret the solution trajectories in terms of pendulum motion.

c) Looking near each equilibrium point, and recalling our classifications of the origin for linear homogenous systems (spiral source, spiral sink, nodal source, nodal sink, saddle, stable center), how would you classify these equilbrium points and characterize their stability?

d) We'll talk about the general linearization procedure that explains the classifications in c, (and works for autonomous systems of more than two first order DEs, where there aren't such accessible pictures),after we do a populations example.
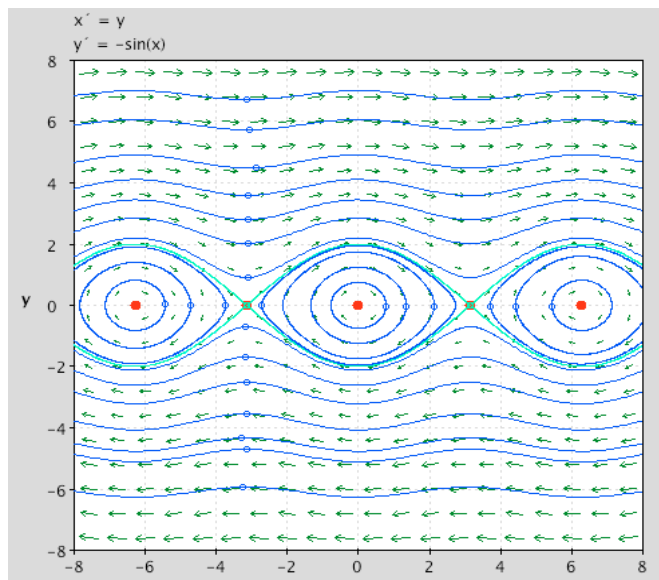
For reference, here are the precise definitions we're using:

The general (non-linear) system of two first order differential equations for $x(t), y(t)$ can be written as
$$x'(t) = F(x(t), y(t), t)$$
$$y'(t) = G(x(t), y(t), t)$$
which we often abbreviate, by writing
$$x' = F(x, y, t)$$
$$y' = G(x, y, t) .$$
If the rates of change F, G only depend on the values of $x(t), y(t)$ but not on t, i.e.
$$x' = F(x, y)$$
$$y' = G(x, y)$$
then the system is called <u>autonomous.</u> Autonomous systems of first order DEs are the focus of Chapter 6, and are the generalization of one autonomous first order DE, as we studied in Chapter 2. In Chapter 6 the text restricts to systems of two equations as above, although most of the ideas generalize to more complicated autonomous systems with three or more interacting functions.

<u>Constant solutions</u> to an autonomous differential equation or system of DEs are called <u>equilibrium solutions.</u> Thus, equilibrium solutions $x(t) \equiv x_*, y(t) \equiv y_*$ have identically zero derivative and will correspond to solutions $[x_*, y_*]^T$ of the nonlinear algebraic system
$$F(x, y) = 0$$
$$G(x, y) = 0$$

- Equilibrium solutions $[x_*, y_*]^T$ to first order autonomous systems
$$x' = F(x, y)$$
$$y' = G(x, y)$$
are called <u>stable</u> if solutions to IVPs starting close (enough) to $[x_*, y_*]^T$ stay as close as desired.
- Equilibrium solutions are <u>unstable</u> if they are not stable.
- Equilibrium solutions $[x_*, y_*]^T$ are called <u>asymptotically stable</u> if they are stable and furthermore, IVP solutions that start close enough to $[x_*, y_*]^T$ converge to $[x_*, y_*]^T$ as $t \to \infty$.
(Notice these definitions are completely analogous to our discussion in Chapter 2.)

Example 2) Consider the "competing species" model from 9.2, shown below. For example and in appropriate units, $x(t)$ might be a squirrel population and $y(t)$ might be a rabbit population, competing on the same island sanctuary.

$$x'(t) = 14\,x - 2\,x^2 - x\,y$$
$$y'(t) = 16\,y - 2\,y^2 - x\,y.$$

2a) Notice that if either population is missing, the other population satisfies a logistic DE. Discuss how the signs of third terms on the right sides of these DEs indicate that the populations are competing with each other (rather than, for example, acting in symbiosis, or so that one of them is a predator of the other). Hint: to understand why this model is plausible for $x(t)$ consider the normalized birth rate rate $\dfrac{x'(t)}{x(t)}$, as we did in Chapter 2.

2b) Find the four equilibrium solutions to this competition model, algebraically.
2c) What does the phase portrait below indicate about the dynamics of this system?
2d) Based on our work in Chapter 5, how would you classify each of the four equilibrium points, including stability?

$x' = 14x-2x^2-xy$
$y' = 16y-2y^2-xy$

8.4741, 10.928
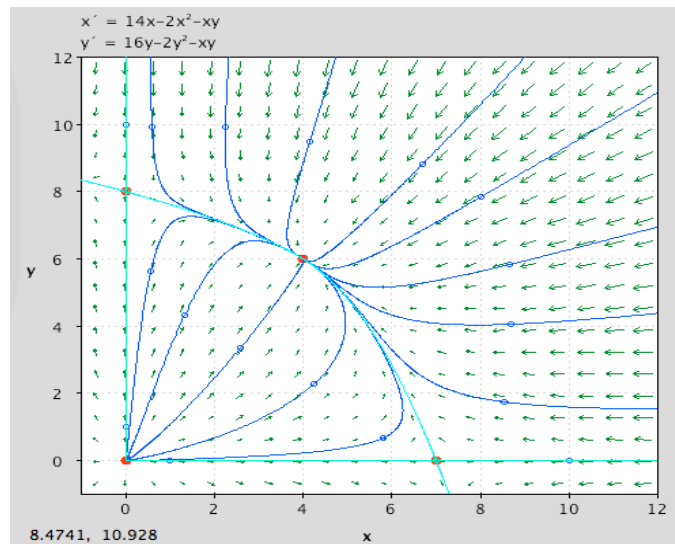
Linearization near equilibrium solutions is a recurring theme in differential equations and in this Math 2280 course. (You may have forgotten, but the "linear drag" velocity model, Newton's law of cooling, and the damped spring equation were all linearizations!!) It's important to understand how to linearize in general, because the linearized differential equations can often be used to understand stability and solution behavior near the equilibrium point, for the original differential equations. Today we'll talk about linearizing systems systems of DE's, which we've not done before in this class.

An easy case of linearization in Example 2 is near the equilbrium solution $\left[x_*, y_*\right]^T = [0, 0]^T$. It's pretty clear that our population system

$$x'(t) = 14\,x - 2\,x^2 - x\,y$$
$$y'(t) = 16\,y - 2\,y^2 - x\,y$$

linearizes to

$$x'(t) = 14\,x$$
$$y'(t) = 16\,y$$

i.e.

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

The eigenvalues are the diagonal entries, and the eigenvectors are the standard basis vectors, so

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 e^{14\,t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{16\,t} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Notice how the phase portrait for the linearized system looks like that for the non-linear system, near the origin:
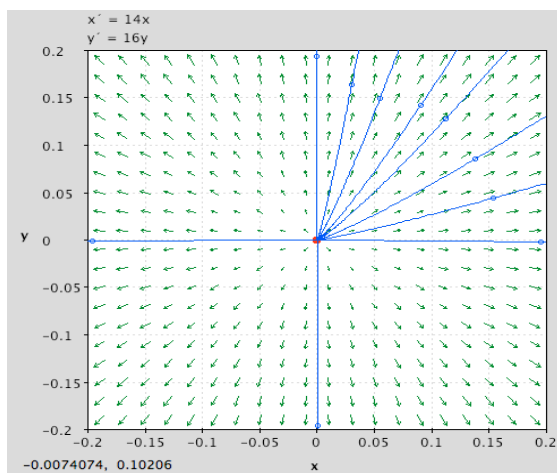
How to linearize with multivariable Calculus: (This would work for systems of n autonomous first order differential equations, but we focus on $n = 2$ in this chapter. Notice how we're not assuming the equilbrium point is the origin. Here's the general system:

$$x'(t) = F(x, y)$$
$$y'(t) = G(x, y)$$

Let $x(t) \equiv x_*, y(t) \equiv y_*$ be an equilibrium solution, i.e.

$$F(x_*, y_*) = 0$$
$$G(x_*, y_*) = 0 \ .$$

For solutions $[x(t), y(t)]^T$ to the original system, define the deviations from equilibrium $u(t), v(t)$ by

$$u(t) := x(t) - x_*$$
$$v(t) := y(t) - y_* \ .$$

Equivalently,

$$x(t) := x_* + u(t)$$
$$y(t) := y_* + v(t)$$

Thus

$$u' = x' = F(x, y) = F(x_* + u, y_* + v)$$
$$v' = y' = G(x, y) = G(x_* + u, y_* + v) \ .$$

Using partial derivatives, which measure rates of change in the coordinate directions, we can approximate

$$u' = F(x_* + u, y_* + v) = F(x_*, y_*) + \frac{\partial F}{\partial x}(x_*, y_*) u + \frac{\partial F}{\partial y}(x_*, y_*) v + \varepsilon_1(u, v)$$

$$v' = G(x_* + u, y_* + v) = G(x_*, y_*) + \frac{\partial G}{\partial x}(x_*, y_*) u + \frac{\partial G}{\partial y}(x_*, y_*) v + \varepsilon_2(u, v)$$

For differentiable functions, the error terms $\varepsilon_1, \varepsilon_2$ shrink more quickly than the linear terms, as $u, v \to 0$. Also, note that $F(x_*, y_*) = G(x_*, y_*) = 0$ because (x_*, y_*) is an equilibrium point. Thus the linearized system that approximates the non-linear system for $u(t), v(t)$, is (written in matrix vector form as):

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x}(x_*, y_*) & \frac{\partial F}{\partial y}(x_*, y_*) \\ \frac{\partial G}{\partial x}(x_*, y_*) & \frac{\partial G}{\partial y}(x_*, y_*) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \ .$$

The matrix of partial derivatives is called the Jacobian matrix for the vector-valued function $[F(x, y), G(x, y)]^T$, evaluated at the point (x_*, y_*). Notice that it is evaluated at the equilibrium point.

People often use the subscript notation for partial derivatives to save writing, e.g F_x for $\frac{\partial F}{\partial x}$ and F_y for $\frac{\partial F}{\partial y}$.

Example 3) We will linearize the rabbit-squirrel (competition) model of the previous example, near the equilibrium solution $[4, 6]^T$. For convenience, here is that system:

$$x'(t) = 14x - 2x^2 - xy$$
$$y'(t) = 16y - 2y^2 - xy$$

3a) Use the Jacobian matrix method of linearizing they system at $[4, 6]^T$. In other words, as on the previous page, set

$$u(t) = x(t) - 4$$
$$v(t) = y(t) - 6$$

So, $u(t), v(t)$ are the deviations of $x(t), y(t)$ from 4, 6, respectively. Then use the Jacobian matrix computation to verify that the linearized system of differential equations that $u(t), v(t)$ approximately satisfy is

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -8 & -4 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

3b) The matrix in the linear system of DE's above has approximate eigendata:

$$\lambda_1 \approx -4.7, \quad \underline{v}_1 \approx [.79, -.64]^T$$
$$\lambda_2 \approx -15.3, \quad \underline{v}_2 \approx [.49, .89]^T$$

We can use the eigendata above to write down the general solution to the homogeneous (linearized) system, to make a rough sketch of the solution trajectories to the linearized problem near $[u, v]^T = [0, 0]^T$, and to classify the equilibrium solution using the Chapter 5 cases. Let's do that and then compare our work to the pplane output on the next page. As we'd expect, phase portrait for the linearized problem near $[u, v]^T = [0, 0]^T$ looks very much like the phase portrait for $[x, y]^T$ near $[4, 6]^T$. This is sensible, since the correspondence between (x, y) and (u, v) involves a translation of $x - y$ coordinate axes to $u - v$ coordinate axes, via the formula.

$$x = u + 4$$
$$y = v + 6$$

Linearization allows us to approximate and understand solutions to non-linear problems near equilbria:

The non-linear problem and representative solution curves:


$x' = 14x-2x^2-xy$
$y' = 16y-2y^2-xy$

9.9458, 5.0979
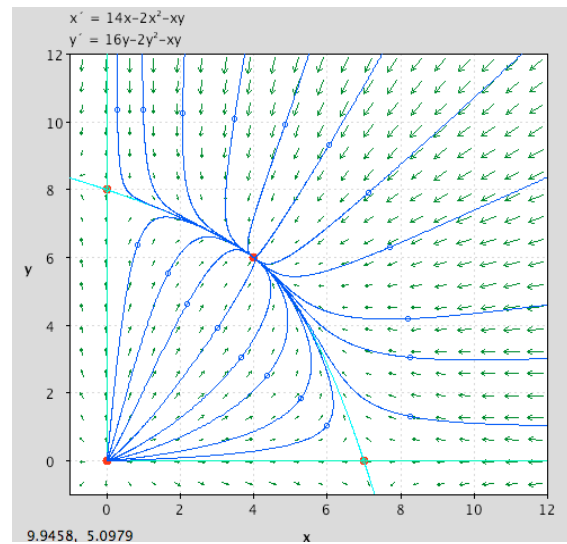
pplane will do the eigenvalue-eigenvector linearization computation for you, if you use the "find an equilibrium solution" option under the "solution" menu item.

Equilbrium Point:
There is a nodal sink at (4, 6)
Jacobian:
$$\begin{matrix} -8 & -4 \\ -6 & -12 \end{matrix}$$
The eigenvalues and eigenvectors are:
-4.7085   (0.77218, -0.63541)
-15.292   (0.48097, 0.87674)

The solutions to the linearized system near $[u, v]^T = [0, 0]^T$ are close to the exact solutions for non-linear deviations, so under the translation of coordinates $u = x - x_*$, $v = y - y_*$ the phase portrait for the linearized system looks like the phase portrait for the non-linear system.
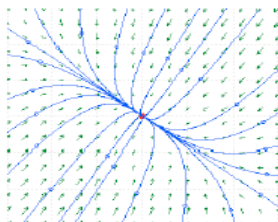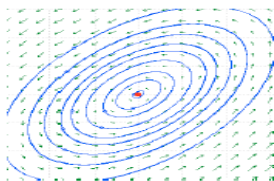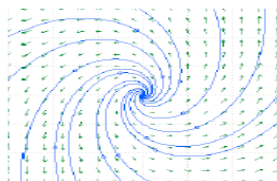

nodal sink
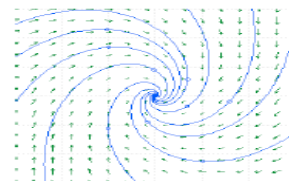$\lambda_1, \lambda_2 < 0$
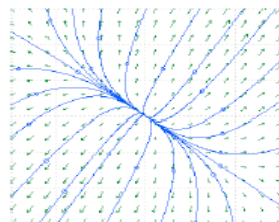
nodal source
$\lambda_1, \lambda_2 > 0$
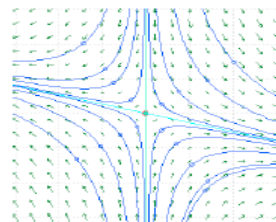
saddle point
$\lambda_1 < 0 < \lambda_2$

center
$Re(\lambda)=0$

spiral source
$Re(\lambda)>0$

spiral sink
$Re(\lambda)<0$

<u>Theorem:</u> Let $[x_*, y_*]$ be an equilibrium point for a first order autonomous system of differential equations.

(i) If the linearized system of differential equations at $[x_*, y_*]$ has real eigendata, and either of an (asymptotically stable) nodal sink, an (unstable) nodal source, or an (unstable) saddle point, then the equilibrium solution for the non-linear system inherits the same stability and geometric properties as the linearized solutions.

(ii) If the linearized system has complex eigendata, and if $\Re(\lambda) \neq 0$, then the equilibrium solution for the non-linear system is also either an (unstable) spiral source or a (stable) spiral sink. If the linearization yields a (stable) center, then further work is needed to deduce stability properties for the nonlinear system.

<u>Example 4</u> Returning to the non-linear pendulum

$$x'(t) = y$$

$$y'(t) = -\frac{g}{L}\sin(x).$$

The solution trajectories ("orbits") follow level curves of the total energy function, which we repeat from page 1, recalling that $x(t) = \theta(t), y(t) = \theta'(t)$,

$$TE(t) = \frac{1}{2}m(Ly)^2 + mgL(1 - \cos(x))$$
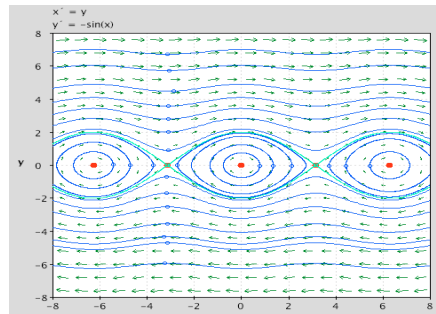
If we compute the Jacobian matrix for this system, we get

$$J(x, y) = \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{vmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{L}\cos(x) & 0 \end{bmatrix}.$$

- When $x = n\pi$ with n even (and $y = 0$),

$$J = \begin{vmatrix} 0 & 1 \\ -\dfrac{g}{L} & 0 \end{vmatrix}$$

the eigenvalues are $\lambda = \pm i\sqrt{\dfrac{g}{L}}$, so for the linearization we have a stable center, but this is the borderline case for the non-linerar problem. Luckly these equilibrium points are exactly where the total energy function has its strict minimum value (of zero), and if a trajectory starts nearby its total energy is almost zero and the trajectory cannot wander away from the equilibrium point - so these are stable centers for the non-linear pendulum.

- When $x = n\pi$ with n odd (and $y = 0$),

$$J = \begin{bmatrix} 0 & 1 \\ +\dfrac{g}{L} & 0 \end{bmatrix}$$

the eigenvalues are $\lambda = \pm\sqrt{\dfrac{g}{L}}$, so for the linearization and the non-linear system we have an unstable saddle!

Example 5) Consider the slightly damped pendulum with $\theta(t)$ with $\dfrac{g}{L} = 1$ and satisfying

$$\theta''(t) + .2\,\theta'(t) + \sin(\theta) = 0$$

so that $[\theta(t), \theta'(t)]^T$ satisfies

$$x'(t) = y$$
$$y'(t) = -\sin(x) - 0.2\,y$$

One can check that we get the same equilibrium points as before, corresponding to the pendulum at rest vertically. The points $(x, y) = (n\pi, 0)$ with n odd are still saddles, but when n is even the stable centers are replaced with spiral sinks. This is an "underdamped" pendulum!
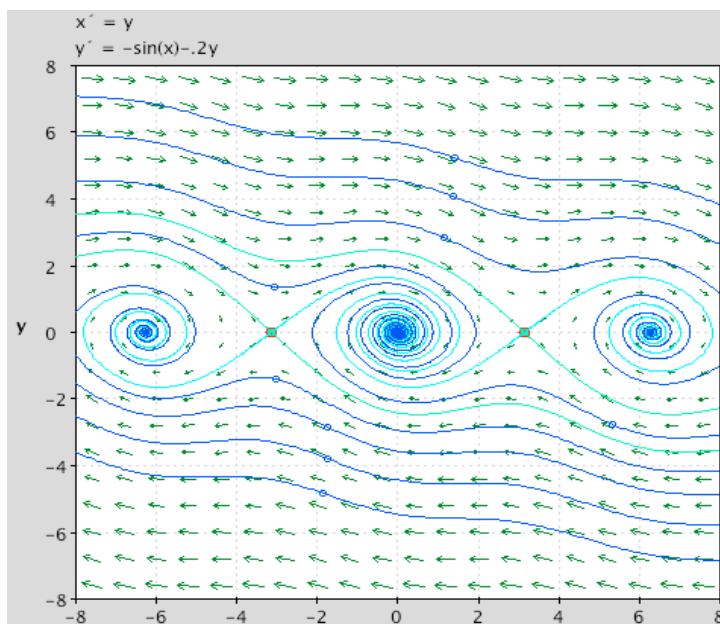
There are lots of interesting population models in section 9.2. Here's another competition model that looks deceptively like Example 2, except the competition got too intense (compare coefficients between the two systems).

Example 6)

$$x'(t) = 14\,x - x^2 - 2\,x\,y$$
$$y'(t) = 16\,y - y^2 - 2\,x\,y$$

Do populations peacefully co-exist in this competition model? A little competition may be healthy, but too much maybe not so much. :-)