Math 2280-001 Week 11
Mar 27-29 (Exam 2 on Mar 31)

Mon Mar 27: Use last Friday's notes to discuss matrix exponentials

Wed Mar 29: 5.6-5.7 Matrix exponentials, linear systems, and variation of parameters for inhomogeneous systems.

Recall: For the first order system

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

- $\Phi(t)$ is a fundamental matrix (FM) if its n columns are a basis for the solution space to the first order system above (i.e. $\Phi(t)$ is the Wronskian matrix for a basis to the solution space).

- $\Phi(t)$ is an FM if and only if $\Phi'(t) = A\,\Phi$ and $\Phi(0)$ is invertible.

- e^{tA} is the unique matrix solution to

$$X'(t) = A\,X$$
$$X(0) = I$$

and may be computed either of two ways:

$$e^{tA} = \Phi(t)\Phi(0)^{-1}$$

where $\Phi(t)$ is any other FM, or via the infinite series

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots + \frac{t^k}{k!}A^k + \ldots$$

Example 1: We showed that if

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \ldots & 0 \\ 0 & \lambda_2 & 0 \ldots & 0 \\ : & : & & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

then

$$e^{t\Lambda} = \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2}{2!}\lambda_1^2 + \ldots & 0 & 0 \ \ldots & 0 \\ 0 & 1 + t\lambda_2 + \frac{t^2}{2!}\lambda_2^2 + \ldots & 0 \ldots & 0 \\ : & : & & : \\ 0 & 0 & \ldots & 1 + t\lambda_n + \frac{t^2}{2!}\lambda_n^2 + \ldots \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} e^{t\lambda_1} & 0 & \dots & 0 \\ 0 & e^{t\lambda_2} & \dots & 0 \\ : & : & : & : \\ 0 & 0 & 0 & e^{t\lambda_n} \end{bmatrix}$$

<u>Example 2)</u> If A is diagonalizable we showed that we got the same answer for e^{tA} using

$$A = S \Lambda S^{-1}$$

and the Taylor series method, as we did we did using the $\Phi(t)\Phi(0)^{-1}$ method. In fact, $\Phi(t) = S\, e^{t\Lambda}$ and $\Phi(0)^{-1} = S^{-1}$, so this makes sense.

How to compute e^{At} when A is not diagonalizable. This method depends on the fundamental fact about how the generalized eigenspaces of a matrix fit together.

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"Recall" (this is really linear algebra material, but most of you haven't seen it.)

For $A_{n\times n}$ let the characteristic polynomial $p(\lambda)=det(A-\lambda I)$ factor as

$$p(\lambda)=(-1)^n(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\dots(\lambda-\lambda_m)^{k_m}$$

Any eigenspace of A for which $dim\left(E_{\lambda_j}\right)<k_j$ is called defective. If A has any defective eigenspaces then it is not diagonalizable. (If none of the eigenspaces are defective, then my amalgamating bases for each eigenspace one obtains a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$, in the case of complex eigendata). However, the larger generalized eigenspaces G_{λ_j} defined by

$$E_{\lambda_j}\subseteq G_{\lambda_j}:=nullspace\left(\left(A-\lambda_j I\right)^{k_j}\right)$$

do always have dimension k_j. If bases for each G_{λ_j} are amalgamated they will form a basis for $\mathbb{R}^n$ or $\mathbb{C}^n$.

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For each basis vector $\underline{\boldsymbol{v}}$ of G_{λ_j} one can construct a basis solution $x(t)$ to $x'=Ax$ as follows:

$$\underline{\boldsymbol{x}}(t)=e^{At}\underline{\boldsymbol{v}}=e^{\lambda_j I t}e^{(A-\lambda_j I)t}\underline{\boldsymbol{v}}$$

$$=e^{\lambda_j t}I\left(I+t\left(A-\lambda_j I\right)+\frac{t^2}{2!}\left(A-\lambda_j I\right)^2+\dots\right)\underline{\boldsymbol{v}}$$

$$=e^{\lambda_j t}\left(\underline{\boldsymbol{v}}+t\left(A-\lambda_j I\right)\underline{\boldsymbol{v}}+\frac{t^2}{2!}\left(A-\lambda_j I\right)^2\underline{\boldsymbol{v}}+\dots\right)$$

$$=e^{\lambda_j t}\left(\underline{\boldsymbol{v}}+t\left(A-\lambda_j I\right)\underline{\boldsymbol{v}}+\frac{t^2}{2!}\left(A-\lambda_j I\right)^2\underline{\boldsymbol{v}}+\frac{t^{k_j-1}}{(k_j-1)!}\left(A-\lambda_j I\right)^{k_j-1}\underline{\boldsymbol{v}}+0+0+\dots\right)$$

$$\underline{\boldsymbol{x}}(t)=e^{\lambda_j t}\left(\underline{\boldsymbol{v}}+t\left(A-\lambda_j I\right)\underline{\boldsymbol{v}}+\frac{t^2}{2!}\left(A-\lambda_j I\right)^2\underline{\boldsymbol{v}}+\frac{t^{k_j-1}}{(k_j-1)!}\left(A-\lambda_j I\right)^{k_j-1}\underline{\boldsymbol{v}}\right).$$

Notes: The final sum is a finite sum because $\underline{\boldsymbol{v}}\in nullspace\left(\left(A-\lambda_j I\right)^{k_j}\right)$! If $\underline{\boldsymbol{v}}$ was an eigenvector, you've just reconstructed

$$\underline{\boldsymbol{x}}(t)=e^{\lambda_j t}\underline{\boldsymbol{v}}$$

Use the n independent solutions found this way to construct a $\Phi(t)$, and compute $e^{At}=\Phi(t)\Phi(0)^{-1}$.

Exercise 1 Find e^{At} for the matrix in the system:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}=\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

tech check:

```
> with(LinearAlgebra) :
  A := [[3, -1], [1, 1]];
  factor(Determinant(A − λ·IdentityMatrix(2)));
```

$$A := \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$(\lambda - 2)^2 \quad \textbf{(1)}$$

So the only eigenvalue is $\lambda = 2$.

```
> B := A − 2· IdentityMatrix(2);
  NullSpace(B);
```

$$B := \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \textbf{(2)}$$

but $E_{\lambda=2}$ is only one-dimensional. However, the generalized eigenspace $G_{\lambda=2} = nullspace(A - 2\,I)^2$ will be two dimensional:

```
> NullSpace(B^2);
```

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \quad \textbf{(3)}$$

Use this generalized nullspace basis to construct a basis of solutions to $x' = A\,x$ and use the resulting $\Phi(t)$ to construct the matrix exponential...

```
> x1 := t → e^(2·t)·(IdentityMatrix(2) + t·B).[1, 0] :
  x1(t);
  x2 := t → e^(2·t)·(IdentityMatrix(2) + t·B + t^2/2·B^2).[0, 1] :
  x2(t);
  Φ := t → ⟨x1(t)|x2(t)⟩ :
  Φ(t);
  Φ(t).Φ(0)^(−1);
  MatrixExponential(t·A);   #these last two should be the same!! In fact, the last three in this case
```

$$\begin{bmatrix} e^{2t}(1+t) \\ t\,e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} -t\,e^{2t} \\ (1-t)\,e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} e^{2t}\,(1+t) & -t\,e^{2t} \\ t\,e^{2t} & (1-t)\,e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} e^{2t}\,(1+t) & -t\,e^{2t} \\ t\,e^{2t} & (1-t)\,e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} e^{2t}\,(1+t) & -t\,e^{2t} \\ t\,e^{2t} & -e^{2t}\,(-1+t) \end{bmatrix} \tag{4}$$

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Variation of parameters: This is what fundamental matrices and matrix exponentials are especially good for....they let you solve non-homogeneous systems without guessing. Consider the non-homogeneous first order system

$$\underline{\mathbf{x}}'(t) = P(t)\underline{\mathbf{x}} + \underline{\boldsymbol{f}}(t) \quad *$$

Let $\Phi(t)$ be an FM for the homogeneous system

$$\underline{\mathbf{x}}'(t) = P(t)\underline{\mathbf{x}}.$$

Since $\Phi(t)$ is invertible for all t we may do a change of functions for the non-homogeneous system:

$$\underline{\mathbf{x}}(t) = \Phi(t)\underline{\mathbf{u}}(t)$$

plug into the non-homogeneous system (*):

$$\Phi'(t)\underline{\mathbf{u}}(t) + \Phi(t)\underline{\mathbf{u}}'(t) = P(t)\Phi(t)\underline{\mathbf{u}}(t) + \underline{\boldsymbol{f}}(t).$$

Since $\Phi' = P\,\Phi$ the first terms on each side cancel each other and we are left with

$$\Phi(t)\underline{\mathbf{u}}'(t) = \underline{\boldsymbol{f}}(t)$$

$$\underline{\mathbf{u}}' = \Phi^{-1}\underline{\boldsymbol{f}}$$

which we can integrate to find a $\underline{\mathbf{u}}(t)$, hence an $\underline{\mathbf{x}}(t) = \Phi(t)\underline{\mathbf{u}}(t)$.

Remark: This is where the (mysterious at the time) formula for variation of parameters in n^{th} order linear DE's came from....

"Recall" (February 24 notes):

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Variation of Parameters: The advantage of this method is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L, and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x),...y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = 0\,.$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \,...\, + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x),...u_n(x)$ have derivatives satisfying the Wronskian matrix equation

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ : \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ : \\ f \end{bmatrix}$$

....

But if we convert the n^{th} order DE into a first order system for $x_1 = y, x_2 = y'$ etc. we have

$$\begin{aligned} &x_1 \quad (= y) \\ &x_1' = x_2 \quad (= y') \\ &x_2' = x_3 \quad (= y'') \\ &x_{n-1}' = x_n \quad \left(= y^{(n-1)}\right) \\ &x_n' = \left(= y^{(n)}\right) = -p_0(x)y_1 - p_1(x)y_2 - \ldots - p_{n-1}(x)y_{n-1} + f. \end{aligned}$$

And each basis solution $y(t)$ for $L(y) = 0$ gives a solution $\left[y, y', y'', \ldots y^{(n-1)}\right]^T$ to the homogeneous system

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & 1 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -p_0 & -p_1 & -p_2 & \ldots & -p_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f \end{bmatrix}.$$

So the original Wronskian matrix for the n^{th} order linear homogeneous DE is a FM for the system above, so the formula we learned in Chapter 3 is a special case of the easier to understand one for first order systems that we just derived, namely

$$\Phi(t)\underline{\boldsymbol{u}}'(t) = \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{u}}' = \Phi^{-1}\underline{\boldsymbol{f}}$$

Returning to first order systems, if we want to solve an IVP for a first order system rather than find the complete general solution, then the following two ways are appropriate:

<u>1) If you want to solve the IVP</u>

$$\underline{\boldsymbol{x}}'(t) = P(t)\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

The the solution will be of the form $\underline{\boldsymbol{x}} = \Phi\,\underline{\boldsymbol{u}}$ (where $\underline{\boldsymbol{u}}' = \Phi^{-1}\underline{\boldsymbol{f}}$ as above). Thus

$$\underline{\boldsymbol{x}}_0 = \Phi(0)\underline{\boldsymbol{u}}_0$$

so

$$\underline{\boldsymbol{u}}_0 = \Phi(0)^{-1}\underline{\boldsymbol{x}}_0.$$

Thus

$$\underline{\boldsymbol{u}}(t) = \underline{\boldsymbol{u}}_0 + \int_0^t \underline{\boldsymbol{u}}'(s)\,\mathrm{d}s$$

$$\underline{\boldsymbol{u}}(t) = \underline{\boldsymbol{u}}_0 + \int_0^t \underline{\boldsymbol{\Phi}}^{-1}(s)\underline{\boldsymbol{f}}(s)\,\mathrm{d}s.$$

Then

$$\underline{\boldsymbol{x}}(t) = \Phi(t)\underline{\boldsymbol{u}}(t)$$
$$\underline{\boldsymbol{x}}(t) = \Phi(t)\left(\underline{\boldsymbol{u}}_0 + \int_0^t \underline{\boldsymbol{\Phi}}^{-1}(s)\underline{\boldsymbol{f}}(s)\,\mathrm{d}s\right).$$

<u>2) If you want to solve the special case IVP</u>

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0$$

where A is a constant matrix, you may derive a special case of the solution formula above just as we did in Chapter 1. This is sort of amazing!

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$
$$\mathrm{e}^{-tA}(\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}}) = \mathrm{e}^{-tA}\underline{\boldsymbol{f}}(t)$$
$$\frac{d}{dt}\left(\mathrm{e}^{-tA}\underline{\boldsymbol{x}}(t)\right) = \mathrm{e}^{-tA}\underline{\boldsymbol{f}}(t).$$

Integrate from 0 to t:

$$\mathrm{e}^{-tA}\underline{\boldsymbol{x}}(t) - \underline{\boldsymbol{x}}_0 = \int_0^t \mathrm{e}^{-sA}\underline{\boldsymbol{f}}(s)\,\mathrm{d}s$$

Move the $\underline{\boldsymbol{x}}_0$ over and multiply both sides by e^{tA}:

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{tA}\left(\underline{\boldsymbol{x}}_0 + \int_0^t \mathrm{e}^{-sA}\underline{\boldsymbol{f}}(s)\,\mathrm{d}s\right).$$

Exercise 2 Consider the non-homogeneous problem related to the homogeneous system in Exercise 1:

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solve this system using the formula

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{t\,A}\left(\underline{\boldsymbol{x}}_0 + \int_0^t \mathrm{e}^{-s\,A}\underline{\boldsymbol{f}}(s)\,\mathrm{d}s\right)$$

(One could also try undetermined coefficients, but variation of parameters requires no "guessing.")

Tech check: (The commands are sort of strange, but might help in your homework.)

```
> with(LinearAlgebra) :
> A := [[3, -1], [1, 1]] :
  MatrixExponential(t·A);
  f := t→[t, 0] :
  x0 := [0, 0] :
```

$$\begin{bmatrix} \mathrm{e}^{2\,t}\,(1+t) & -t\,\mathrm{e}^{2\,t} \\ t\,\mathrm{e}^{2\,t} & -\mathrm{e}^{2\,t}\,(-1+t) \end{bmatrix} \qquad \mathbf{(5)}$$

```
> integrand := s→simplify(MatrixExponential(−s·A).f(s)) :  #integrand in formula above
> integrand(t);  #checking
```

$$\begin{bmatrix} -e^{-2t}(-1+t)\,t \\ -t^2 e^{-2t} \end{bmatrix} \tag{6}$$

```
> integrated := unapply(map(int, integrand(s), s = 0 .. t), t) :  #"map"  applies a function to each entry of an array...
      # "unapply" makes a function out of output
```

```
> integrated(t);  #checking
```

$$\begin{bmatrix} \frac{1}{2} t^2 e^{-2t} \\ -\frac{1}{4} + \frac{1}{4} e^{-2t} + \frac{1}{2} t e^{-2t} + \frac{1}{2} t^2 e^{-2t} \end{bmatrix} \tag{7}$$

```
> x := unapply(simplify(MatrixExponential(t·A).(x0 + integrated(t))), t) :
  x(t);  #checking answer
```

$$\begin{bmatrix} \frac{1}{4} t \left(e^{2t} - 1\right) \\ \frac{1}{4} e^{2t}(-1+t) + \frac{1}{4} t + \frac{1}{4} \end{bmatrix} \tag{8}$$

```
> with(DEtools) :
  dsolve({x1'(t) = 3·x1(t) −x2(t) + t, x2'(t) = x1(t) + x2(t), x1(0) = 0, x2(0) = 0});
```

$$\left\{x1(t) = \frac{1}{4} t e^{2t} - \frac{1}{4} t, x2(t) = -\frac{1}{4} e^{2t} + \frac{1}{4} + \frac{1}{4} t + \frac{1}{4} t e^{2t}\right\} \tag{9}$$

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