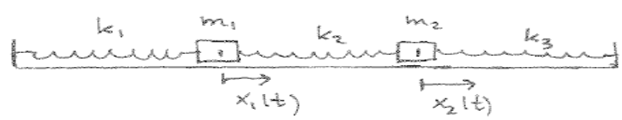
Math 2280-001
Week 10: Mar 20-24, sections 5.4, 5.6
questions on homework due tomorrow?

5.4 Mass-spring systems: untethered mass-spring trains, and forced oscillation non-homogeneous problems.

Consider the mass-spring system below, with no damping. Although we draw the picture horizontally, it would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field.

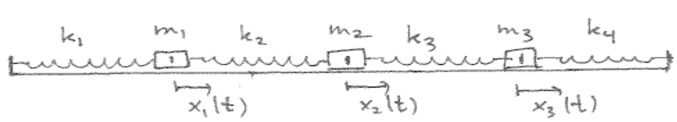


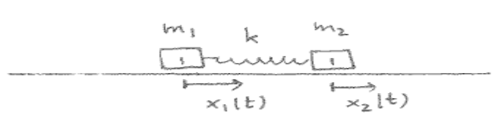
Let's make sure we understand why the natural system of DEs and IVP for this system is

$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2$$
$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 1a) What is the dimension of the solution space to this homogeneous linear system of differential equations? Why? (Hint: after deriving the system of second order differential equations write down an equivalent system of first order differential equations.)

1b) What if one had a configuration of n masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why? Examples:





We can write the system of DEs for the system at the top of the previous page in matrix-vector form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We denote the diagonal matrix on the left as the "mass matrix" M, and the matrix on the right as the spring constant matrix K (although to be completely in sync with Chapter 5 it would be better to call the spring matrix $-K$). All of these configurations of masses in series with springs can be written as

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2 + k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}.$$

(You can think of A as the "acceleration" matrix.)
Notice that the simplification above is mathematically identical to the algebraic operation of multiplying the first matrix equation by the (diagonal) inverse of the diagonal mass matrix M. In all cases:

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}} \quad \Rightarrow \underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}\ , \text{with } A = M^{-1}K.$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}} \ .$$

Based on our previous experiences, the natural thing for this homogeneous system of linear differential equations is to try and find a basis of solutions of the form

$$\underline{\boldsymbol{x}}(t) = e^{r\,t}\underline{\boldsymbol{v}}$$

We would maybe also think about first converting the second order system to an equivalent first order system of twice as many DE's, one for for each position function and one for each velocity function. But let's try the substitution directly, in analogy to what we did for higher order single linear differential equations back in Chapter 3.

Now, in the present case of systems of masses and springs we are assuming there is no damping. Thus, the total energy - consisting of the sum of kinetic and potential energy - will always be conserved. Any two complex solutions of the form

$$\underline{\boldsymbol{x}}(t) = e^{r\,t}\boldsymbol{v}^{\pm} = e^{(a\,\pm\,\omega i)t}\boldsymbol{v}^{\pm}$$

would yield two real solutions $\underline{\boldsymbol{X}}(t), \underline{\boldsymbol{Y}}(t)$ where

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{X}}(t) \pm i\,\underline{\boldsymbol{Y}}(t) \ .$$

Because of conservation of energy ($TE = KE + PE$ must be constant), neither $\underline{\boldsymbol{X}}(t)$ nor $\underline{\boldsymbol{Y}}(t)$ can grow or decay exponentially - if a solution grew exponentially the total energy would also grow exponentially; if it decayed exponentially the total energy would decay exponentially. SO, we must have $a = 0$. In other words, in order for the total energy to remain constant we must actually have

$$\underline{\boldsymbol{x}}(t) = e^{i\,\omega\,t}\underline{\boldsymbol{v}}\,.$$

Substituting this $x(t)$ into the homogeneous DE

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$$

yields the necessary condition

$$-\omega^2 e^{i\,\omega\,t}\underline{\boldsymbol{v}} = e^{i\,\omega\,t}A\,\underline{\boldsymbol{v}} \quad .$$

So $\underline{\boldsymbol{v}}$ must be an eigenvector, with non-positive eigenvalue $\lambda = -\omega^2$,

$$A\,\underline{\boldsymbol{v}} = -\omega^2\underline{\boldsymbol{v}}\,.$$

And since row reduction will find real eigenvectors for real eigenvalues, we can find eigenvectors $\underline{\boldsymbol{v}}$ with real entries. And the two complex solutions

$$\underline{\boldsymbol{x}}(t) = e^{\pm\,i\omega t}\underline{\boldsymbol{v}} = \cos(\omega\,t)\underline{\boldsymbol{v}} \pm i\sin(\omega\,t)\underline{\boldsymbol{v}}$$

yield the two real solutions

$$\underline{\boldsymbol{X}}(t) = \cos(\omega\,t)\underline{\boldsymbol{v}}, \qquad \underline{\boldsymbol{Y}}(t) = \sin(\omega\,t)\underline{\boldsymbol{v}}\,.$$

So, we skip the exponential solutions altogether, and go directly to finding homogeneous solutions of the form above. We just have to be careful to remember that $\underline{\boldsymbol{v}}$ is an eigenvector with eigenvalue $\lambda = -\omega^2$, i.e. $\omega = \sqrt{-\lambda}\,.$

Note: In analaogy with the scalar undamped oscillator DE

$$x''(t) + \omega_0^2\, x = 0$$

where we could read off and check the solutions

$$\cos\left(\omega_0 t\right),\ \sin\left(\omega_0 t\right)$$

directly without going throught the characteristic polynomial, it is easy to check that

$$\cos(\omega\, t)\underline{\boldsymbol{v}},\ \sin(\omega\, t)\underline{\boldsymbol{v}}$$

each solve the conserved energy mass spring system

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$$

as long as

$$-\omega^2\,\underline{\boldsymbol{v}} = A\,\underline{\boldsymbol{v}}.$$

This leads to the

Solution space algorithm: Consider a very special case of a homogeneous system of linear differential equations,

$$\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}\ .$$

If $A_{n\times n}$ is a diagonalizable matrix and if all of its eigenvalues are negative, then for each eigenpair $\left(\lambda_j, \underline{\boldsymbol{v}}_j\right)$ there are two linearly independent solutions to $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$ given by

$$\boldsymbol{x}_j(t) = \cos\left(\omega_j\, t\right)\underline{\boldsymbol{v}}_j \qquad \underline{\boldsymbol{y}}_j(t) = \sin\left(\omega_j\, t\right)\underline{\boldsymbol{v}}_j$$

with

$$\omega_j = \sqrt{-\lambda_j}\ .$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\boldsymbol{x}}''(t) = A\,\underline{\boldsymbol{x}}$, i.e. a basis for the solution space.

Remark: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $\left(c_1 + c_2\, t\right)\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

a) Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

b) Deduce the eigendata for the acceleration matrix A which is $\frac{k}{m}$ times this matrix.

c) Find the $4-$ dimensional solution space to this two-mass, three-spring system.

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3\,k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cos(\omega_1 t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\omega_2 t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

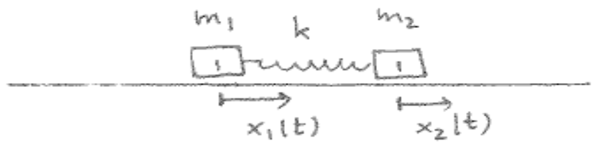
$$= \left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos(\omega_2 t) + c_4 \sin(\omega_2 t)\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is four dimensional.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

Exercise 4) Consider a train with two cars connected by a spring:



4a) Derive the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero)

4b). Find the eigenvalues and eigenvectors. Then find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\mathbf{v}}$ verify that you get two solutions

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{v}} \text{ and } \underline{\mathbf{x}}(t) = t\,\underline{\mathbf{v}},$$

rather than the expected $\cos(\omega\, t)\underline{\mathbf{v}},\ \sin(\omega\, t)\underline{\mathbf{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems.

Math 2280-001
Wed Mar 22

5.4 Mass-spring systems and forced oscillation non-homogeneous problems.

• Finish Monday's notes if necessary, about unforced, undamped oscillations in multi mass-spring configurations. As a check of your understanding between first order systems and second order conservative mass-spring systems, see if you can answer the exercise below. Then proceed to experiment and forced oscillations on following pages.

Summary exercise: Here are two systems of differential equations, and the eigendata is as shown. The first order system could arise from an input-output model, and the second one could arise from an undamped two mass, three spring model. Write down the general solution to each system.

1a)

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1b)

$$\begin{bmatrix} x_1{}'' \\ x_2{}'' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eigendata: For the matrix

$$\begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

for the eigenvalue $\lambda = -5$, $\underline{\mathbf{v}} = [-2, 1]^T$ is an eigenvector; for the eigenvalue $\lambda = -1$, $\underline{\mathbf{v}} = [2, 1]^T$ is an eigenvector

The two mass, three spring system....Experiment!

Data: Each mass is 50 grams. Each spring mass is 10 grams. (Remember, and this is a defect, our model assumes massless springs.) The springs are "identical", and a mass of 50 grams stretches the spring 15.6 centimeters. (We should recheck this since it's old data; we should also test the spring's "Hookesiness"). With the old numbers we get Hooke's constant

```
> Digits := 4:
> solve(k·.156 = .05·9.806, k)
```

$$3.143 \qquad (1)$$

```
>
```

Here's Maple confirmation for some of our work yesterday:

```
> with(LinearAlgebra):
  A := Matrix(2, 2, [-2·k/m, k/m, k/m, -2·k/m]);
  Eigenvectors(A);
```

$$A := \begin{bmatrix} -\frac{2\,k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2\,k}{m} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3\,k}{m} \\ -\frac{k}{m} \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad (2)$$

Predict the two natural periods from the model and our experimental value of k, m. Then make the system vibrate in each mode individually and compare your prediction to the actual periods of these two fundamental modes.

ANSWER: If you do the model correctly and my office data is close to our class data, you will come up with theoretical natural periods of close to .46 and .79 seconds. I predict that the actual natural periods are a little longer, especially for the slow mode. (In my office experiment I got periods of 0.482 and 0.855 seconds.) What happened?

EXPLANATION: The springs actually have mass, equal to 10 grams each. This is almost on the same order of magnitude as the yellow masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more-complicated situation, using the ideas of total energy discussed in section 3.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator http://www.math.utah.edu/~korevaar/2280spring15/feb25.pdf, assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an *A*-matrix the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{6\,k}{6\,m + 5\,m_s}$$

$$\lambda_2 = -\frac{6\,k}{2\,m + m_s}.$$

(Hints: the "M" matrix is not diagonal but the "K" matrix is the same.)

If you use these values, then you get period predictions

```
> m := .05;
  ms := .010;
  k := 3.143;
  Omega1 := sqrt(6*k/(6*m + 5*ms));
  Omega2 := sqrt(6*k/(2*m + ms));
  T1 := evalf(2*Pi/Omega1);
  T2 := evalf(2*Pi/Omega2);
```

$$m := 0.05$$
$$ms := 0.010$$
$$k := 3.143$$
$$\Omega 1 := 7.340$$
$$\Omega 2 := 13.09$$
$$T1 := 0.8559$$
$$T2 := 0.4801 \qquad \textbf{(3)}$$

of .856 and .480 seconds per cycle. Is that closer?

Forced oscillations (still undamped):

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}} + \underline{\boldsymbol{F}}(t)$$
$$\Rightarrow \underline{\mathbf{x}}''(t) = A\underline{\mathbf{x}} + M^{-1}\underline{\boldsymbol{F}}(t)\ .$$

If the forcing is sinusoidal,

$$M\underline{\mathbf{x}}''(t) = K\underline{\mathbf{x}} + \cos(\omega t)\underline{\boldsymbol{G}_0}$$
$$\Rightarrow\ \underline{\mathbf{x}}''(t) = A\underline{\mathbf{x}} + \cos(\omega t)\underline{\boldsymbol{F}_0}$$

with $\underline{\boldsymbol{F}_0} = M^{-1}\underline{\boldsymbol{G}_0}$.

From the fundamental theorem for linear transformations we know that the general solution to this inhomogeneous linear problem is of the form

$$\underline{\mathbf{x}}(t) = \underline{\mathbf{x}}_P(t) + \underline{\mathbf{x}}_H(t)\ ,$$

and we've been discussing how to find the homogeneous solutions $\underline{\mathbf{x}}_H(t)$.

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\underline{\mathbf{x}}_P(t) = \cos(\omega t)\ \underline{\boldsymbol{c}}$$

where the vector $\underline{\boldsymbol{c}}$ is what we need to find.

Exercise 2) Substitute the guess $\underline{\mathbf{x}}_P(t) = \cos(\omega t)\ \underline{\boldsymbol{c}}$ into the DE system

$$\underline{\mathbf{x}}''(t) = A\underline{\mathbf{x}} + \cos(\omega t)\underline{\boldsymbol{F}_0}$$

to find a matrix algebra formula for $\underline{\boldsymbol{c}} = \underline{\boldsymbol{c}}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

Solution:

$$\underline{\boldsymbol{c}}(\omega) = -\left(A + \omega^2 I\right)^{-1}\underline{\boldsymbol{F}_0}\ .$$

Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue.

Exercise 3) Continuing with the configuration from Monday's notes, but now for an inhomogeneous forced problem, let $k = m$, and force the second mass sinusoidally:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We know from previous work that the natural frequencies are $\omega_1 = 1$, $\omega_2 = \sqrt{3}$ and that

$$\underline{\mathbf{x}}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\, t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\underline{\mathbf{x}}_P(t)$, as on the preceding page. Notice that this steady periodic solution blows up as $\omega \to 1$ or $\omega \to \sqrt{3}$. (If we don't have time to work this by hand, we may skip directly to the technology check on the next page. But since we have quick formulas for inverses of 2 by 2 matrices, this is definitely a computation we could do by hand.)

Solution: As long as $\omega \neq 1, \sqrt{3}$, the general solution $\underline{\mathbf{x}} = \underline{\mathbf{x}}_P + \underline{\mathbf{x}}_H$ is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t) \begin{bmatrix} \dfrac{3}{(\omega^2 - 1)(\omega^2 - 3)} \\ \dfrac{6 - 3\omega^2}{(\omega^2 - 1)(\omega^2 - 3)} \end{bmatrix}$$

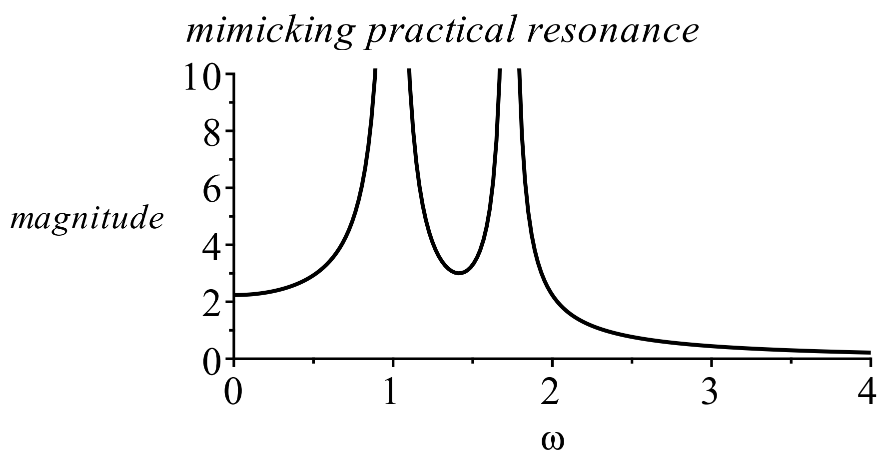
$$+ \ C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\, t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Interpretation as far as inferred practical resonance for slightly damped problems: If there was even a small amount of damping, the homogeneous solution would actually be transient (it would be exponentially decaying and oscillating - underdamped). There would still be a sinusoidal particular solution, which would have a formula close to our particular solution, the first term above, as long as $\omega \neq 1, \sqrt{3}$. (There would also be a relatively smaller $\sin(\omega t)\underline{\mathbf{d}}$ term as well.) So we can infer the practical resonance behavior for different ω values with slight damping, by looking at the size of the $\underline{\mathbf{c}}(\omega)$ term for the undamped problem....see next page for visualizations.

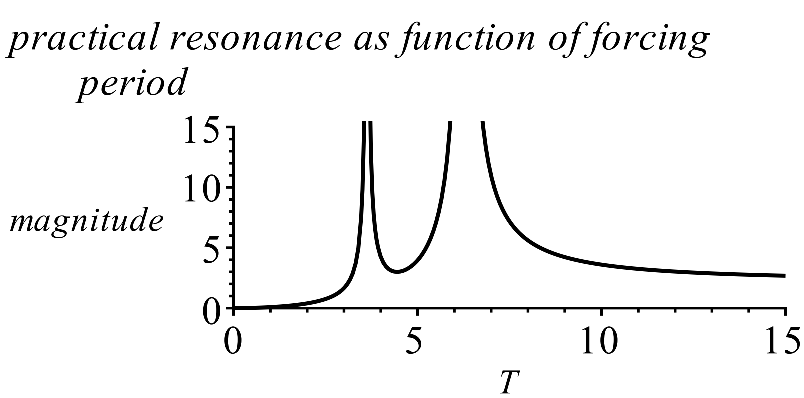
```
> restart :
> with(LinearAlgebra) :
> A := Matrix(2, 2, [-2, 1, 1, -2]) :
> F0 := Vector([0, 3]) :
> Iden := IdentityMatrix(2) :
> c := ω → (A + ω^2 · Iden)^(-1) . (-F0) :  # the formula we worked out by hand
> c(ω);
```

$$\begin{bmatrix} \dfrac{3}{\omega^4 - 4\,\omega^2 + 3} \\ -\dfrac{3\left(\omega^2 - 2\right)}{\omega^4 - 4\,\omega^2 + 3} \end{bmatrix} \qquad \textbf{(4)}$$

```
> with(plots) :
  with(LinearAlgebra) :
> plot(Norm(c(ω), 2), ω = 0 ..4, magnitude = 0 ..10, color = black, title = `mimicking practical resonance`);
     # Norm(c(ω),2) is the magnitude of the c(ω) vector
```



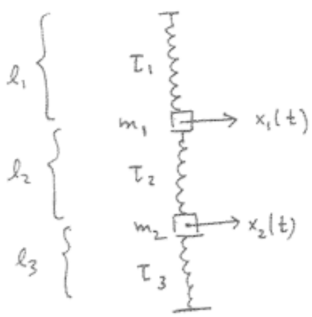

```
> plot(Norm(c(2·Pi/T), 2), T = 0 ..15, magnitude = 0 ..15, color = black, title
     = `practical resonance as function of forcing period`);
```




```
>
```

There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings:

- Transverse oscillations! (i.e. directions ⊥ to the mass-spring configuration)



τ_1, τ_2, τ_3 are the tensions (pulling forces) of the stretched springs

By linearization, a good model would be

$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1+K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = K_2(x_1 - x_2) - K_3 x_2 = K_2 x_1 \quad -(K_2+K_3)x_2$$

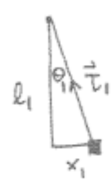
where $K_1, K_2\, K_3$ are positive constants as before

→ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2^{nd} order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:

[force picture, e.g.



horiz force from top spring on mass 1

$$= -\tau_1 \sin\theta_1 = -\tau_1 \frac{x_1}{\sqrt{\ell_1^2 + x_1^2}} \approx -\tau_1 \frac{x_1}{\ell_1} = -\frac{\tau_1}{\ell_1} x_1$$

So $K_1 = \frac{\tau_1}{\ell_1}$

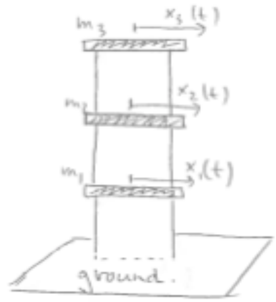
similarly, $K_2 = \frac{\tau_2}{\ell_2}$, $K_3 = \frac{\tau_3}{\ell_3}$

for our physics demo springs, equilibrium length ≈ 0, very Hookesian so $\tau \doteq k\ell$; $\frac{\tau}{\ell} \approx k$, so actually almost recover same fundamental modes!!]

- An interesting shake-table demonstration:

http://www.youtube.com/watch?v=M_x2jOKAhZM

Below is a discussion of how to model the unforced "three-story" building shown shaking in the video above, from which we can see which modes will be excited. There is also a "two-story" building model in the video, and its matrix and eigendata follow. Here's a schematic of the three-story building:



For the unforced (homogeneous) problem, the accelerations of the three massive floors (the top one is the roof) above ground and of mass m, are given by

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Note the -1 value in the last diagonal entry of the matrix. This is because $x_3(t)$ is measuring displacements for the top floor (roof), which has nothing above it. The "k" is just the linearization proportionality factor, and depends on the tension in the walls, and the height between floors, etc, as discussed on the previous page.

Here is eigendata for the unscaled matrix $\left(\frac{k}{m}=1\right)$. For the scaled matrix you'd have the same eigenvectors, but the eigenvalues would all be multiplied by the scaling factor $\frac{k}{m}$ and the natural frequencies would all be scaled by $\sqrt{\frac{k}{m}}$. Symmetric matrices like ours (i.e matrix equals its transpose) are always diagonalizable with real eigenvalues and eigenvectors...and you can choose the eigenvectors to be mutually perpendicular. This is called the "Spectral Theorem for symmetric matrices" and is an important fact in many math and science applications...you can read about it here: http://en.wikipedia.org/wiki/Symmetric_matrix.) If we tell Maple that our matrix is symmetric it will not confuse us with unsimplified numbers and vectors that may look complex rather than real.

```
> with(LinearAlgebra) :
> A := Matrix(3, 3, [−2.0, 1, 0, 1, −2, 1, 0, 1, −1]);
      # I used at least one decimal value so Maple would evaluate in floating point
```

$$A := \begin{bmatrix} -2.0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \textbf{(5)}$$

```
> Digits := 5 : # 5 digits should be fine, for our decimal approximations.
```

```
> eigendata := Eigenvectors(Matrix(A, shape = symmetric)) : # to take advantage of the
                  # spectral theorem
  lambdas := eigendata[1] :   #eigenvalues
  evectors := eigendata[2] :   #corresponding eigenvectors - for fundamental modes
  omegas := map(sqrt, -lambdas);  # natural angular frequencies
  f := x -> 2·evalf(Pi)/x :
  periods := map(f, omegas); #natural periods
  eigenvectors := map(evalf, evectors); #  get digits down to 5
```

$$omegas := \begin{bmatrix} 1.8019 \\ 1.2470 \\ 0.44504 \end{bmatrix}$$

$$periods := \begin{bmatrix} 3.4870 \\ 5.0386 \\ 14.118 \end{bmatrix}$$

$$eigenvectors := \begin{bmatrix} -0.59101 & -0.73698 & 0.32799 \\ 0.73698 & -0.32799 & 0.59101 \\ -0.32799 & 0.59101 & 0.73698 \end{bmatrix} \quad \textbf{(6)}$$

```
> B := Matrix(2, 2, [-2, 1.0, 1, -1]) :
  eigendata := Eigenvectors(Matrix(B, shape = symmetric)) : # to take advantage of the
                  # spectral theorem
  lambdas := eigendata[1] :   #eigenvalues
  evectors := eigendata[2] :   #corresponding eigenvectors - for fundamental modes
  omegas := map(sqrt, -lambdas);  # natural angular frequencies
  f := x -> 2·evalf(Pi)/x :
  periods := map(f, omegas); #natural periods
  eigenvectors := map(evalf, evectors); #  get digits down to 5
```

$$omegas := \begin{bmatrix} 1.6180 \\ 0.61804 \end{bmatrix}$$

$$periods := \begin{bmatrix} 3.8834 \\ 10.166 \end{bmatrix}$$

$$eigenvectors := \begin{bmatrix} -0.85065 & -0.52573 \\ 0.52573 & -0.85065 \end{bmatrix} \quad \textbf{(7)}$$

```
>
```

Exercise 4) Interpret the data above, in terms of the natural modes for the shaking building . In the youtube video the first mode to appear is the slow and dangerous "sloshing mode", where all three floors oscillate in phase, with amplitude ratios 33 : 59 : 74 from the first to the third floor. What's the second mode that gets excited? The third mode? (They don't show the third mode in the video.)

Remark) All of the ideas we've discussed in section 5.4 also apply to molecular vibrations. The eigendata in these cases is related to the "spectrum" of light frequencies that correspond to the natural fundamental modes for molecular vibrations.

Math 2280-001
Fri Mar 24

5.6 Matrix exponentials and linear systems: The analogy between first order systems of linear differential equations (Chapter 5) and scalar linear differential equations (Chapter 1) is much stronger than you may have expected. This will become especially clear on Monday, when we study section 5.7.

Definition Consider the linear system of differential equations for $x(t)$:

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

where $A_{n\times n}$ is a constant matrix as usual. If $\left\{\underline{\boldsymbol{x}}_1(t), \underline{\boldsymbol{x}}_2(t), \ldots \underline{\boldsymbol{x}}_n(t)\right\}$ is a basis for the solution space to this system, then the matrix having these solutions as columns,

$$\Phi(t) := \left[\underline{\boldsymbol{x}}_1(t) \middle| \underline{\boldsymbol{x}}_2(t) \middle| \cdots \middle| \underline{\boldsymbol{x}}_n(t)\right]$$

is called a Fundamental Matrix (FM) to this system of differential equations. Notice that this equivalent to saying that $X(t) = \Phi(t)$ solves

$$\begin{cases} X'(\mathrm{t}) = A\,X \\ X(0) \qquad nonsingular\ (i.e.\ invertible) \end{cases}$$

(just look column by column). Notice that a FM is just the Wronskian matrix for a solution space basis.

Example 1 page 351

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5)$$

$\lambda = -2$:

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \Rightarrow \underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\lambda = 5$:

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \Rightarrow \underline{\boldsymbol{v}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

general solution

$$\underline{\boldsymbol{x}}(t) = c_1 e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Possible FM:

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

general solution:

$$\Phi(t)\underline{\boldsymbol{c}} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Theorem: If $\Phi(t)$ is a FM for the first order system $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$ then the solution to

$$IVP\ \begin{cases} \underline{\boldsymbol{x}}'(\mathrm{t}) = A\,\underline{\boldsymbol{x}} \\ \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \end{cases}$$

is

$$\underline{\boldsymbol{x}}(t) = \Phi(t)\Phi(0)^{-1}\underline{\boldsymbol{x}}_0$$

proof: Since $\underline{\boldsymbol{x}}(t) = \Phi(t)\Phi(0)^{-1}\underline{\boldsymbol{x}}_0 = \Phi(t)\left[\Phi(0)^{-1}\underline{\boldsymbol{x}}_0\right]$ is a linear combination of the columns of $\Phi(t)$ it is a solution to the homogeneous DE $\underline{\boldsymbol{x}}'(\mathrm{t}) = A\,\underline{\boldsymbol{x}}$. Its value at $t = 0$ is
$\underline{\boldsymbol{x}}(0) = \Phi(0)\Phi(0)^{-1}\underline{\boldsymbol{x}}_0 = \left[\Phi(0)\Phi(0)^{-1}\right]\underline{\boldsymbol{x}}_0 = I\,\underline{\boldsymbol{x}}_0 = \underline{\boldsymbol{x}}_0$.

□

Exercise 1) Continuing with the example on page 1, use the formula above to solve

$$IVP\ \begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

ans: $\underline{\boldsymbol{x}}(t) = \frac{3}{7}\mathrm{e}^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{2}{7}\mathrm{e}^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Remark: If $\Phi(t)$ is a Fundamental Matrix for $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$ and if C is an invertible matrix of the same size, then $\Phi(t)C$ is also a FM. Check: Does $X(t) = \Phi(t)C$ satisfy

$$\begin{cases} X'(\mathrm{t}) = A\,X \\ X(0) \qquad \textit{nonsingular } (\textit{i.e. invertible}) \end{cases} ?$$

$$\frac{d}{dt}\left(\Phi(t)C\right) = \Phi'(t)C \qquad \text{(universal product rule .... see last page of notes)}$$

$$= (A\,\Phi)C$$
$$= A\,(\Phi C).$$

Also, $X(0) = \Phi(0)C$ is a product of invertible matrices, so is invertible as well. Thus $X(t) = \Phi(t)C$ is an FM.

□

(Notice this argument would not work if we had used $C\Phi(t)$ instead.)

If $\Phi(t)$ is any FM for $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$ then $X(t) = \Phi(t)\Phi(0)^{-1}$ solves

$$\begin{cases} X'(\mathrm{t}) = A\,X \\ X(0) = I \end{cases}.$$

Notice that there is only one matrix solution to this IVP, since the j^{th} column $\underline{\boldsymbol{x}}_j(t)$ is the (unique) solution to

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{e}}_j.$$

Definition The unique FM that solves

$$\begin{cases} X'(\mathrm{t}) = A\,X \\ X(0) = I \end{cases}$$

is called the matrix exponential, e^{tA} ...because:

This generalizes the scalar case. In fact, notice that if we wish to solve

$$IVP \begin{cases} \underline{\boldsymbol{x}}'(\mathrm{t}) = A\,\underline{\boldsymbol{x}} \\ \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \end{cases}$$

the solution is

$$\underline{\boldsymbol{x}}(t) = e^{tA}\underline{\boldsymbol{x}}_0,$$

in analogy with Chapter 1.

Exercise 2) Continuing with our example, for the DE

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

with

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

and FM

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

compute e^{tA}. Check that the solution to the IVP in Exercise 1 is indeed $e^{tA}\underline{\boldsymbol{x}}_0$.

```
> with(LinearAlgebra) :
  A := [[4, 2], [3, -1]] :
  MatrixExponential(t·A);  #check work on previous page
```

$$\begin{bmatrix} \frac{1}{7}e^{-2t} + \frac{6}{7}e^{5t} & \frac{2}{7}e^{5t} - \frac{2}{7}e^{-2t} \\ \frac{3}{7}e^{5t} - \frac{3}{7}e^{-2t} & \frac{6}{7}e^{-2t} + \frac{1}{7}e^{5t} \end{bmatrix} \quad \textbf{(8)}$$

```
>
```

But wait!
Didn't you like how we derived Euler's formula using Taylor series?
Here's an alternate way to think about e^{tA}:

For $A_{n \times n}$ consider the matrix series

$$e^{A} := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots + \frac{1}{k!}A^k + \ldots$$

Convergence: pick a large number M, so that each entry of A satisfies $|a_{ij}| \le M$. Then

$$entry_{ij}\left(A^2\right) \le nM^2$$
$$entry_{ij}\left(A^3\right) \le n^2M^3\ldots$$
$$entry_{ij}\left(A^k\right) \le n^{k-1}M^k$$

so the matrix series converges absolutly in each entry (dominated by the Calc 2 series for the scalar e^{Mn}).

Then define

$$e^{tA} := I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots + \frac{t^k}{k!}A^k + \ldots$$

Notice that for $X(t) = e^{tA}$ defined by the power series above, and assuming the true fact that we may differentiate the series term by term,

$$X'(t) = 0 + A + \frac{2t}{2!}A^2 + \frac{3t^2}{3!}A^3 + \ldots.\frac{kt^{k-1}}{k!}A^k + \ldots$$
$$= A + tA^2 + \frac{t^2}{2!}A^3 + \ldots \frac{t^{k-1}}{(k-1)!}A^k$$
$$= A\left(I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^{(k-1)}}{(k-1)!}A^{k-1} + \ldots\right)$$
$$= A\,X.$$

Also,

$$X(0) = I.$$

Thus, since there is only one matrix function that can satisfy

$$\begin{cases} X'(t) = A\,X \\ X(0) = I \end{cases}$$

we deduce

<u>Theorem</u> The matrix exponential e^{tA} may be computed either of two ways:

$$e^{tA} = \Phi(t)\Phi(0)^{-1}$$
$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots + \frac{t^k}{k!}A^k + \ldots$$

Exercise 3 Let A be a diagonal matrix Λ,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \dots & 0 \\ 0 & \lambda_2 & 0 \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Use the Taylor series definition and the FM definition to verify twice that

$$e^{t\Lambda} = \begin{bmatrix} e^{t\lambda_1} & 0 & \dots & 0 \\ 0 & e^{t\lambda_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & e^{t\lambda_n} \end{bmatrix}$$

Hint: products of diagonal matrices are diagonal, and the diagonal entries multiply, so

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \dots & 0 \\ 0 & \lambda_2^k & 0 \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

<u>Example</u> How to recompute e^{tA} for

$$A := \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

using power series and Math 2270: The similarity matrix made of eigenvectors of A

$$S = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

yields

$$A\,S = S\,\Lambda :$$

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

so

$$A = S\Lambda S^{-1} .$$

Thus $A^k = S\Lambda^k S$ (telescoping product), so

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots + \frac{t^k}{k!}A^k + \ldots \\ &= S\left(I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \ldots + \frac{t^k}{k!}\Lambda^k + \ldots\right)S^{-1} \\ &= S\,e^{t\Lambda}S^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{5t} \end{bmatrix}\frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & -2e^{-2t} \\ 3e^{5t} & e^{5t} \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix} \end{aligned}$$

which agrees with our original computation using the FM.

Three important properties of matrix exponentials:

1) $e^{[0]} = I$, where $[0]$ is the $n \times n$ zero matrix. (Why is this true?)

2) If $AB = BA$ then $e^{A+B} = e^A e^B = e^B e^A$ (but this identity is not generally true when A and B don't commute). (See homework.)

3) $\left(e^A\right)^{-1} = e^{-A}$. (Combine (1) and (2).)

Using these properties there is a "straightforward" algorithm to compute e^{tA} even when A is not diagonalizable (and it doesn't require the use of chains studied in section 5.5). See Theorem 3 in section 5.6 We'll study more details on Monday, but here's an example:

<u>Exercise 4)</u> Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Find e^{tA} by writing $A = D + N$ where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and using $e^{tD + tN} = e^{tD} e^{tN}$. Hint: $N^3 = 0$ so the Taylor series for e^{tN} is very short.

<u>Universal product rule for differentiation:</u> Recall the 1-variable product rule for differentiation for a function of a single variable t, based on the limit definition of derivative. We'll just repeat that discussion, but this time for any product "$*$" that distributes over addition, for scalar, vector, or matrix functions. We also assume that for the product under consideration, scalar multiples s behave according to

$$s(f*g) = (sf)*g = f*(sg).$$

We don't assume that $f*g = g*f$ so must be careful in that regard. Here's how the proof goes:

$$D_t(f*g)(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(f(t+\Delta t)*g(t+\Delta t) - f(t)*g(t)\right).$$

We add and subtract a middle term, to help subsequent algebra:

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(f(t+\Delta t)*g(t+\Delta t) - f(t+\Delta t)*g(t) + f(t+\Delta t)*g(t) - f(t)*g(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(f(t+\Delta t)*g(t+\Delta t) - f(t+\Delta t)*g(t)\right) + \frac{1}{\Delta t}\left(f(t+\Delta t)*g(t) - f(t)*g(t)\right).$$

We assume that multiplication by $*$ distributes over addition:

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} f(t+\Delta t)*\left(g(t+\Delta t) - g(t)\right) + \frac{1}{\Delta t}\left(f(t+\Delta t) - f(t)\right)*g(t) .$$

The sum rule for limits and rearranging the scalar factor let's us rearrange as follows:

$$= \lim_{\Delta t \to 0} f(t+\Delta t)*\left(\frac{1}{\Delta t}\right)\left(g(t+\Delta t) - g(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(f(t+\Delta t) - f(t)\right)*g(t).$$

Differentiable functions are continuous, so we take limits and get:

$$D_t(f*g)(t) = f(t)*g'(t) + f'(t)*g(t) .$$

The proof above applies to

- scalar function times scalar function (Calc I)
- scalar function time vector function (Calc III)
- dot product or cross product of two functions (Calc III)
- scalar function times matrix function (our class)
- matrix function times matrix or vector function (our class)

This proof does not apply to composition $f \circ g(t) := f(g(t))$, because composition does not generally distribute over addition, and this is why we have the chain rule for taking derivatives of composite functions.