~~c) Solve the second order DE IVP for $x(t)$ in order to deduce a solution to the first system order IVP. Use Chapter 3 methods.~~

$$B\vec{v} = \left[\mathrm{col}_1(B) \,\vdots\, \mathrm{col}_2(B)\right]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$$
$$= v_1\,\mathrm{col}_1(B) + v_2\,\mathrm{col}_2(B) = \vec{0}$$

d) Solve the first order system IVP in a using eigenvalue-eigenvector methods. Notice that the matrix "characteristic polynomial" is the same as the Chapter 3 "characteristic polynomial". Check that the $x_1(t)$ you find, i.e. the first component function of the system solution, is actually the $x(t)$ you found in part c.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix}$$
$$= \lambda(\lambda+7)+6 = \lambda^2+7\lambda+6$$
$$= (\lambda+6)(\lambda+1) = 0$$
$$\lambda = -1, -6.$$

$$(A-\lambda I)\vec{v} = \vec{0}$$

$E_{\lambda=-1}$ $\quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -6 & -6 & 0 \end{array}\right]$ $\quad \lambda=-1$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \leftarrow$$

$E_{\lambda=-6}$ $\quad \left[\begin{array}{cc|c} 6 & 1 & 0 \\ -6 & -1 & 0 \end{array}\right]$

$$\vec{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}, \quad e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix} \leftarrow$$

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-5}\begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -10 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

e) Understand the phase portrait below, for the first order system IVP, in terms of the over-damped mass motion of the original second order differential equation. Also notice the influence of the eigenspaces in the solution trajectory!

So, $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

soln to ①, is $x_1(t)$

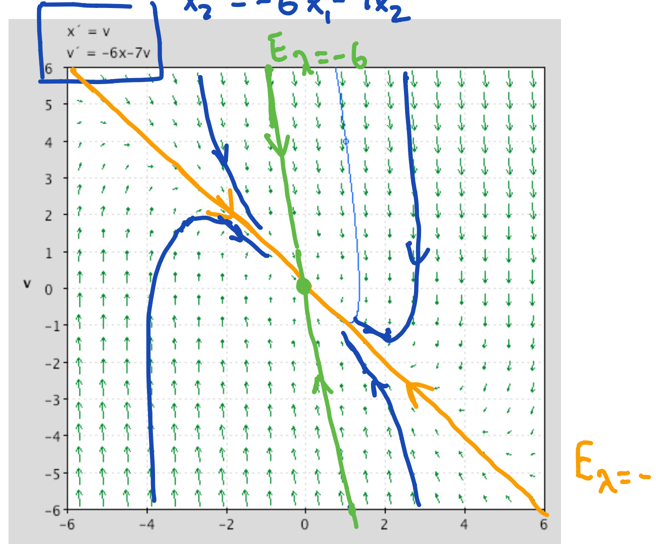
$$x_1(t) = 2e^{-t} - e^{-6t}$$

$$① \begin{cases} x'' + 7x' + 6x = 0 \\ x(0) = 1 \\ x'(0) = 4 \end{cases}$$

$$p(r) = r^2 + 7r + 6$$
$$= (r+6)(r+1)$$

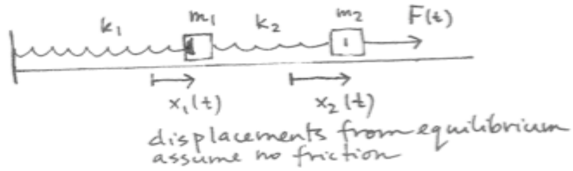
$x_1' = x_2$

$x_2' = -6x_1 - 7x_2$

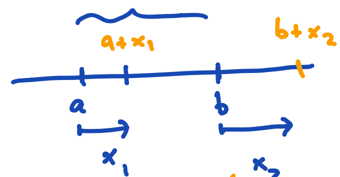


Higher order systems of DE's are also equivalent to first order systems, as illustrated in the next example.

Consider this configuration of two coupled masses and springs:



equil dist $= b-a.$



new dist $= (b+x_2)-(a+x_1)$
$= (b-a)+(x_2-x_1)$

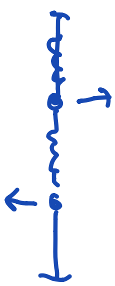
additional separation

Exercise 8) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case?

$$m_1 x_1'' = \text{net forces} = F_{\text{spring }1} + F_{\text{spring }2}$$

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' = -k_2(x_2 - x_1) + F(t) \end{cases}$$

$$\begin{cases} m_1 x_1'' = -(k_1+k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - k_2 x_2 + F(t) \end{cases}$$



OR
(see how)

Exercise 9) Consider the IVP from Exercise 6, with the special values $m_1 = 2, m_2 = 1; k_1 = 4, k_2 = 2; F(t) = 40\sin(3t)$:

$$m_{\underset{2}{1}}\, x_1'' = -\underbrace{(k_1+k_2)}_{6}\, x_1 + \underbrace{k_2}_{2}\, x_2$$

$$m_{\underset{1}{2}}\, x_2'' = \underset{2}{k_2}\, x_1 - \underset{2}{k_2}\, x_2 + \underset{40\sin 3t}{F(t)}$$

1

$$x_1'' = -3x_1 + x_2$$
$$x_2'' = 2x_1 - 2x_2 + 40\sin(3t)$$
$$x_1(0) = b_1,\ x_1'(0) = b_2$$
$$x_2(0) = c_1,\ x_2'(0) = c_2 .$$

9a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define

$$v_1(t) := x_1'(t) \leftarrow$$
$$v_2(t) := x_2'(t) \leftarrow$$

then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

2

$$x_1' = v_1$$
$$x_2' = v_2$$
$$x_1'' = v_1' = -3x_1 + x_2$$
$$x_2'' = v_2' = 2x_1 - 2x_2 + 40\sin(3t)$$
$$x_1(0) = b_1$$
$$v_1(0) = b_2$$
$$x_2(0) = c_1$$
$$v_2(0) = c_2 .$$

soltn to 1
$x_1(t), x_2(t)$

yields soltn to 2
$x_1(t)$
$v_1(t)$
$x_2(t)$
$v_2(t)$

9b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.

soltn to 2 yields soltn to 1

$x_1' = v_1$.
$x_1'' = v_1' = -3x_1 + x_2$

Math 2280-001
Week 9, March 6-10 5.1-5.3
Mon Mar 6
5.1-5.2 Linear systems of differential equations

Monday: Finish Friday's notes
Hw: the §4.1 material due next assignment 3/21
(you'll start w8.4 in class today, & I'll pull up pplane).

- Finish last Friday's notes to understand why any differential equation or system of differential equations can be converted into an equivalent (larger) system to a first order differential equations, and to overview the methods we will be using to solve first order systems of differential equations.

- Then proceed ...

Theorems for linear systems of differential equations:

Theorem 1 For the IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}})$ is continuous in the $t-$variable and differentiable in its $\underline{\boldsymbol{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

"new" today:

Theorem 3) Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{\boldsymbol{x}}(t)$ differentiable on an interval, the operator

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t)$$

is linear, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{z}}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{\boldsymbol{z}}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t)).$$

check!

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{f}}(t)$$

$\forall\ t \in I$ is

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_p(t) + \underline{\boldsymbol{x}}_H(t)$$

where $\underline{\boldsymbol{x}}_p(t)$ is any single particular solution and $\underline{\boldsymbol{x}}_H(t)$ is the general solution to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\mathbf{0}}$$

We frequently write the homogeneous linear system of DE's as

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t)\ .$$

3.3) For $A(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the $t-$interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

is n-dimensional. Here's why:

- Let $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$ defined by

$$\left[W(\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \ldots, \underline{\boldsymbol{X}}_n)\right](t_0) := \left[\underline{\boldsymbol{X}}_1(t_0) | \underline{\boldsymbol{X}}_2(t_0) | \cdots | \underline{\boldsymbol{X}}_n(t_0)\right]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible Wronskian matrix.)

- Then for any $\underline{\boldsymbol{b}} \in \mathbb{R}^n$ the IVP

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{b}}$$

has solution $\underline{\boldsymbol{x}}(t) = c_1\underline{\boldsymbol{X}}_1(t) + c_2\underline{\boldsymbol{X}}_2(t) + \ldots + c_n\underline{\boldsymbol{X}}_n(t)$ where the linear combination coefficients comprise the solution vector to the Wronskian matrix equation

$$\left[\begin{array}{c|c|c|c} \underline{\boldsymbol{X}}_1(t_0) & \underline{\boldsymbol{X}}_2(t_0) & \cdots & \underline{\boldsymbol{X}}_n(t_0) \end{array}\right]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

$[W]\vec{c} = \vec{b}$

$\vec{c} = [W]^{-1}\vec{b}$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$, and since each IVP has only one solution, $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{\boldsymbol{b}} = \underline{\mathbf{0}}$) is the one with $\underline{\boldsymbol{c}} = \underline{\mathbf{0}}$. Thus $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), \ldots \underline{\boldsymbol{X}}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and their number n is the dimension of the solution space.

$y_H: \; p(r) = r^3 + a_2 r^2 + a_1 r + a_0$

$\vec{Z}_H: \; |A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & -a_2-\lambda \end{vmatrix} = -a_0(1) + a_1(-\lambda) + (-a_2-\lambda)(\lambda^2) = -[\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0]$

Remark: If the first order system arises by converting an n^{th} order linear differential equation for $y(x)$ into a linear system of n first order differential equations, then solutions $y(x)$ to the DE correspond to vector solutions $\left[y(x), y'(x), \ldots, y^{(n-1)}(x)\right]$ for the first order systems, so the Chapter 3 Wronskians correspond exactly to the Chapter 5 Wronskians.

Exercise 1) Check this Remark. $n = 3.$

$y''' + a_2 y'' + a_1 y' + a_0 y = f$

$y_H: \; f = 0.$

y_1, y_2, y_3

$W(y_1, y_2, y_3) = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$

$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} y' \\ y'' \\ -a_0 y - a_1 y' - a_2 y'' \end{bmatrix} + \cancel{\begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}} \to 0$ for y_H

$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$

$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}' = \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

Theorem 4) The eigenvalue-eigenvector method for a solution space basis to the homogeneous system (as discussed informally in last week's notes and examples): For the system

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

with $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$, $A_{n \times n}$, if the matrix A is diagonalizable (i.e. there exists a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ of $\mathbb{R}^n$ made out of eigenvectors of A, i.e. $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ for each $j = 1, 2, \ldots, n$), then the functions

$$e^{\lambda_j t}\,\underline{\boldsymbol{v}}_j, \quad j = 1, 2, \ldots n$$

are a basis for the (homogeneous) solution space, i.e. each solution is of the form

$$\underline{\boldsymbol{x}}_H(t) = c_1 e^{\lambda_1 t}\,\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\,\underline{\boldsymbol{v}}_2 + \ldots + c_n e^{\lambda_n t}\,\underline{\boldsymbol{v}}_n\,.$$

sol'ns $\vec{Z}_1(t) = \begin{bmatrix} y_1 \\ y_1' \\ y_1'' \end{bmatrix}$, $\vec{Z}_2 = \begin{bmatrix} y_2 \\ y_2' \\ y_2'' \end{bmatrix}$, $\vec{Z}_3 = \begin{bmatrix} y_3 \\ y_3' \\ y_3'' \end{bmatrix}$

$W(Z_1, Z_2, Z_3) = \left[\vec{Z}_1 \middle| \vec{Z}_2 \middle| \vec{Z}_3 \right]$

proof: check the Wronskian matrix at $t = 0$, its the matrix that has the eigenvectors in its columns, and is invertible because they're a basis for $\mathbb{R}^n$.

• We will continue using the eigenvalue-eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

that we discussed last week.

So far we've not considered the possibility of complex eigenvalues and eigenvectors. Linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc. Then the model space is $\mathbb{C}^n$ rather than $\mathbb{R}^n$.

Definition: $\underline{\boldsymbol{v}} \in \mathbb{C}^n$ ($\underline{\boldsymbol{v}} \neq \underline{\mathbf{0}}$) is a complex eigenvector of the matrix A, with eigenvalue $\lambda \in \mathbb{C}$ if $A\,\underline{\boldsymbol{v}} = \boldsymbol{\lambda}\,\underline{\boldsymbol{v}}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work an example. We can also refer to the general discussion on the following pages, at appropriate stages.

Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

$G' = -.1G - .4H$
$H' = .1G - .1H$

<u>Exercise 1a)</u> Understand why the signs of the matrix entries are reasonable.

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

<u>1b)</u> The first step is to get the eigendata of the matrix. Do this, and compare with the Maple check on the next page.

$$\begin{vmatrix} -.1-\lambda & -.4 \\ .1 & -.1-\lambda \end{vmatrix} = (\lambda+.1)^2 + .04 = 0$$

$(\lambda+.1)^2 = -.04$

$\lambda + .1 = \pm .2i$

$\lambda = -.1 \pm .2i$

$\lambda = -.1 \pm .2i$

$[A - \lambda I]\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -.2i & -.4 & 0 \\ .1 & -.2i & 0 \end{array}\right]$$

$$\begin{array}{l} 10R_2 \\ 5R_1 \end{array} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ -i & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ iR_1+R_2 \end{array} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

$\vec{z}(t) = e^{\lambda t}\vec{v}$

$= e^{(-.1+.2i)t} \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

$= e^{-.1t}\left(\cos(.2t) + i\sin(.2t)\right) \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

$= e^{-.1t} \begin{bmatrix} -2\sin(.2t) \\ \cos(.2t) \end{bmatrix} + i e^{-.1t} \begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix}$

$\vec{Z}(t) = \vec{X}(t) + i\vec{Y}(t)$

$\overline{\vec{Z}}(t) = \vec{X}(t) - i\vec{Y}(t)$

> with(LinearAlgebra) :

> $A := \begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix};$

Eigenvectors(A);

$$A := \begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{10} + \frac{1}{5} \mathrm{I} \\ -\frac{1}{10} - \frac{1}{5} \mathrm{I} \end{bmatrix}, \begin{bmatrix} 2\,\mathrm{I} & -2\,\mathrm{I} \\ 1 & 1 \end{bmatrix} \qquad \textbf{(1)}$$

Notice that Maple writes a capital $I = \sqrt{-1}$.

1c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

$$\vec{Z}(t) = \vec{X}(t) + i\vec{Y}(t).$$

$$\vec{Z}'(t) = A\vec{Z}$$

$$\vec{X}'(t) + i\vec{Y}'(t) = A(\vec{X} + i\vec{Y})$$

$$\vec{X}' + i\vec{Y}' = A\vec{X} + iA\vec{Y}$$

$$\Rightarrow \quad \vec{X}' = A\vec{X}$$
$$\vec{Y}' = A\vec{Y}$$

So $\left\{ e^{-.1t}\begin{bmatrix} -2\sin(.2t) \\ \cos(.2t) \end{bmatrix}, e^{-.1t}\begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix} \right\}$ is a basis of real fcns!

1d) Solve the initial value problem.

$$\begin{bmatrix} G' \\ H' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G \\ H \end{bmatrix} = c_1 e^{-.1t}\begin{bmatrix} -2\sin(.2t) \\ \cos(.2t) \end{bmatrix} + c_2 e^{-.1t}\begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix}$$

@ $t=0$: $\begin{bmatrix} 100 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$\Rightarrow c_1 = 0$$
$$c_2 = 50$$

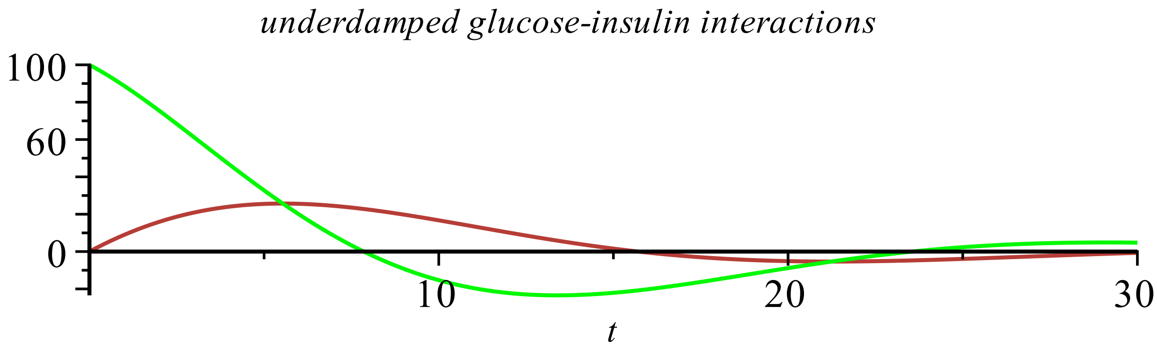
$$\begin{bmatrix} G \\ H \end{bmatrix} = 50e^{-.1t}\begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix}$$

Here are some pictures to help understand what the model is predicting ... you could also construct these graphs using pplane.

(1) Plots of glucose vs. insulin, at time t hours later:

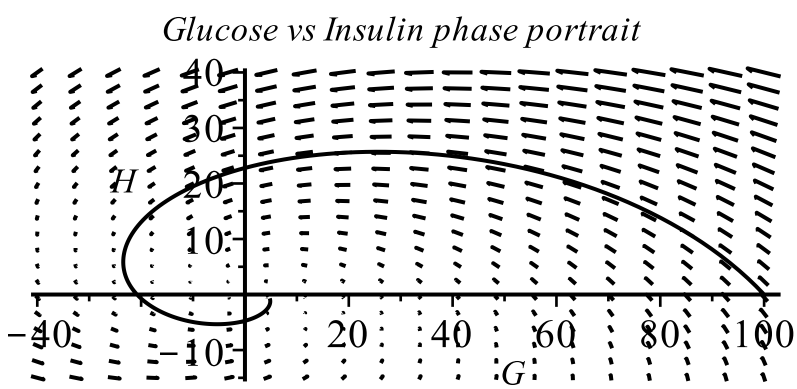
> $with(plots)$:

> $G := t \to 100 \cdot \exp(-.1 \cdot t) \cdot \cos(.2 \cdot t)$:
$H := t \to 50 \cdot \exp(-.1 \cdot t) \cdot \sin(.2 \cdot t)$:
$plot1 := plot(G(t), t = 0 ..30, color = green)$:
$plot2 := plot(H(t), t = 0 ..30, color = brown)$:
$display(\{plot1, plot2\}, title = `underdamped\ glucose\text{-}insulin\ interactions`)$;



2) A phase portrait of the glucose-insulin system:

> $pict1 := fieldplot([-.1 \cdot G - .4 \cdot H, .1 \cdot G - .1 \cdot H], G = -40 ..100, H = -15 ..40)$:
$soltn := plot([G(t), H(t), t = 0 ..30], color = black)$:
$display(\{pict1, soltn\}, title = `Glucose\ vs\ Insulin\ phase\ portrait`)$;



Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \in \mathbb{C}^n$$

satisfy $A\,\underline{\boldsymbol{v}} = \boldsymbol{\lambda}\,\underline{\boldsymbol{v}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ is a complex solution to

$$\underline{\boldsymbol{x}}'(t) = A\,\boldsymbol{x}$$

because $\underline{\boldsymbol{z}}'(t) = \lambda e^{\lambda t}\underline{\boldsymbol{v}}$ and this is equal to $A\,\underline{\boldsymbol{z}} = \boldsymbol{A}\,e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t}A\,\underline{\boldsymbol{v}}$.

- But if we write $\underline{\boldsymbol{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\boldsymbol{z}}(t) = \underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t)$$

then the equality

$$\underline{\boldsymbol{z}}'(t) = A\,\boldsymbol{z}$$
$$\Rightarrow \underline{\boldsymbol{x}}'(t) + i\,\underline{\boldsymbol{y}}'(t) = A(\underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t)) = A\,\underline{\boldsymbol{x}}(t) + i\,A\,\underline{\boldsymbol{y}}(t)\,.$$

Equating the real and imaginary parts on each side yields

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}(t)$$
$$\underline{\boldsymbol{y}}'(t) = A\,\underline{\boldsymbol{y}}(t)$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A\,(\underline{\alpha} + i\,\underline{\boldsymbol{\beta}}) = (a + b\,i)(\underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}})$ then it is straightforward to check that $A\,(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}) = (a - b\,i)(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}})$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i\,b)t}(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}) = \underline{\boldsymbol{x}}(t) - i\,\underline{\boldsymbol{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one).

- More details of what the real soluitions look like:

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \in \mathbb{C}^n$$

$$\Rightarrow \mathrm{e}^{\lambda t}\underline{\boldsymbol{v}} = \mathrm{e}^{a\,t}(\cos(b\,t) + i\sin(b\,t))\cdot(\underline{\boldsymbol{\alpha}} + \underline{\boldsymbol{\beta}}\,i) = \underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t).$$

So the real solutions are

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{a\,t}(\cos(bt)\underline{\boldsymbol{\alpha}} - \sin(bt)\underline{\boldsymbol{\beta}})$$
$$\underline{\boldsymbol{y}}(t) = \mathrm{e}^{a\,t}(\cos(bt)\underline{\boldsymbol{\beta}} + \sin(bt)\underline{\alpha})$$

• The Glucose-insulin example is linearized, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these. Prof. Fred Adler and a recent graduate student Chris Remien in the Math Department, and collaborating with the University Hospital recently modeled liver poisoning by acetominophen (brand name Tyleonol), by studying a non-linear system of 8 first order differential equations. They came up with a state of the art and very useful diagnostic test:

http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

Here's a link to their published paper. For fun, I copied and pasted the non-linear system of first order differential equations from a preprint of their paper, below:

http://onlinelibrary.wiley.com/doi/10.1002/hep.25656/full

http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf

Acetaminophen

APAP $$\frac{dA}{dt} = -\frac{\alpha}{H_{max}} AH - \delta_a A$$

liver toxin

NAPQI $$\frac{dN}{dt} = \frac{qp\alpha}{H_{max}} A - \gamma NG$$

Glutathione removes toxin

GSH $$\frac{dG}{dt} = \kappa - \gamma NG - \delta_g G$$

Functional Hepatocytes $$\frac{dH}{dt} = rH\left(1 - \frac{H+Z}{H_{max}}\right) - \eta NH$$

Damaged Hepatocytes $$\frac{dZ}{dt} = \eta NH - \delta_z Z$$

liver enzymes

AST $$\frac{dS}{dt} = \frac{d_z \beta_s}{\theta H_{max}} Z - \delta_s (S - S_{min})$$

ALT $$\frac{dL}{dt} = \frac{d_z \beta_l}{\theta H_{max}} Z - \delta_l (L - L_{min})$$

Clotting Factor $$\frac{dF}{dt} = \beta_f \left(\frac{H}{H_{max}} - F\right)$$

measure these
& "INR" measure
of liver damage
w.r.t. clotting factors
at admission

to recover info about
1) size of overdose
2) how long before admission