$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -9 \\ -18 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

@ $t=0$: $\begin{bmatrix} 9 \\ 0 \end{bmatrix}$ ✓

long way.

$$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_1 = \frac{1}{2}t$

$v_2 = t$

$\vec{v} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

Exercise 4) Lessons learned from Exercise 3:

a) What condition on the constant matrix $A_{n \times n}$ will allow you to solve every initial value problem

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \in \mathbb{R}^n$$

using the method in Exercise 3 ? Hint: Math 2270 discussions (If that condition fails there are other ways to find the unique solutions.)

short way.

$A\vec{v} = v_1\,col_1(A) + v_2\,col_2(A) + \ldots + v_n\,col_n(A)$

$1\cdot col_1(A) + 2\,col_2(A) = \vec{0}$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{u}(t) = e^{0t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}(t) = e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

are a basis for the sol'n space to $\vec{x}' = A\vec{x}$ is #3.

(1) span: Can solve any IVP with $\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$

at $t=0$: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

and each IVP only has one sol'n (any sol'n has initial vector)

so every sol't. is a linear combo.

b) What is the dimension of the solution space to the first order system

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

when $\underline{\boldsymbol{x}}_0 \in \mathbb{R}^n$ and $A = A_{n \times n}$?

(2) independence.

$c_1\vec{u}(t) + c_2\vec{v}(t) \equiv 0$

@ $t=0$. $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = 0$.

④ $\vec{x}'(t) = A\vec{x}$ $A_{n\times n}$ constant matrix

If there's a basis of $\mathbb{R}^n$, $\mathbb{C}^n$, $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ made out of eigenvectors of A

$A\vec{v}_j = \lambda_j\vec{v}_j$.

then $\vec{x}_H(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + \ldots + c_ne^{\lambda_n t}\vec{v}_n$

to solve $\vec{x}(0) = \vec{b}$

$\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$

$$\vec{b} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Converting higher order DE's or systems of DE's to equivalent first order systems of DEs:

It is always the case that the natural initial value problems for single differential equations or systems of differential equations are equivalent to initial value problems for larger systems of first order differential equations. The following discussion will illustrate this equivalence.

For example, consider this second order underdamped IVP for $x(t)$:

$$x'' + 2x' + 5x = 0$$
$$x(0) = 4 \qquad ①$$
$$x'(0) = -4 \ .$$

Exercise 5)

5a) Suppose that $x(t)$ solves the IVP above. Define $x_1(t) := x(t)$ and $x_2(t) := x'(t)$. Show that $[x_1, x_2]^T$ solves the first order system initial value problem

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad ②$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}.$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -5x-2x' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

②

5b) Conversely, show that if $[x_1, x_2]^T$ solves the first order system in 5a, then its first component function $x(t) := x_1(t)$ solves the original second order differential equation initial value problem. Thus, there is an equivalence between the original second order DE initial value problem, and the corresponding IVP for the related system of first order DE's.

$$x_1' = x_2 \qquad \longrightarrow \qquad \frac{d}{dt}: \ x_1'' = x_2' = -5x_1 - 2x_2$$
$$x_2' = -5x_1 - 2x_2$$
$$x_1'' = -5x_1 - 2x_1'$$
$$x_1'' + 2x_1' - 5x_1 = 0$$
$$x_1(0) = 4$$
$$\text{since } x_2 = x_1', \quad x_1'(0) = -4$$

<u>5c)</u> Solve the second order IVP in order to deduce a solution to the first order IVP in <u>5a</u>. (Use Chapter 3 techniques.)

<u>5d)</u> How does the Chapter 3 "characteristic polynomial" in the second order differential equation compare with (Math 2270) eigenvalue "characteristic polynomial" for the matrix in the first order system 5<u>a</u>? hmmm. What if you used complex eigenvalues and eigenvectors to solve the first order system IVP. Could you recover the solution $x(t)$ to the original second order DE IVP?

<u>5d)</u> Is your analytic solution $\left[x_1(t), x_2(t)\right] = \left[x(t), v(t)\right]$ in <u>5a</u> consistent with the parametric curve shown below, in a "pplane" screenshot? The picture is called a "phase portrait" for position and velocity.

5c) $x'' + 2x' + 5x = 0$

$x(0) = 4$

$x'(0) = -4$

$\boxed{p(r) = r^2 + 2r + 5} = (r+1)^2 + 4 = 0$

$(r+1)^2 = -4$

$r + 1 = \pm 2i$

$r = -1 \pm 2i$

$x(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

$x(0) = 4 = c_1$

$x'(0) = -4 = -c_1 + 2c_2$

$\Rightarrow c_1 = 4,\ c_2 = 0.$

$x'(t) = c_1\left(-e^{-t}\cos 2t - 2e^{-t}\sin 2t\right) + c_2\left(-e^{-t}\sin 2t + 2e^{-t}\cos 2t\right)$

$\boxed{x(t) = 4e^{-t}\cos 2t}$ ①

So sol'n to 1st order system ② is

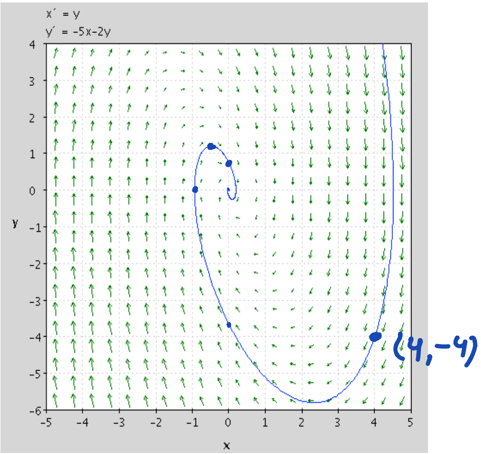
$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} 4e^{-t}\cos 2t \\ -4e^{-t}\cos 2t - 8e^{-t}\sin 2t \end{bmatrix}$$

$$= 4e^{-t}\begin{bmatrix} 1 & 0 \\ -1 & +2 \end{bmatrix}\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}$$

↑ clockwise circle

ellipse

spiral converging exponentially to $\vec{0}$



5d): $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

look for a basis of sol'ns $e^{\lambda t}\vec{v}$

$A\vec{v} = \lambda\vec{v}$.

$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = \lambda(\lambda+2) + 5$

$\boxed{= \lambda^2 + 2\lambda + 5} = 0$

$|\lambda I - A|$

$\boxed{\lambda = -1 \pm 2i}$

$e^{\lambda t}\vec{v}$?? & $e^{\bar{\lambda} t}\bar{\vec{v}}$

Exercise 6)

consider this second order ~~underdamped~~ overdamped IVP for $x(t)$:

$$\cancel{x'' + 6x' + 7x = 0}$$

$$x'' + 7x' + 6x = 0$$

$$x(0) = 1$$
$$x'(0) = 4 \ .$$

①

a) Show without finding a formula for the solution $x(t)$, that whatever the function is, then by defining $x_1(t) := x(t)$ and $x_2(t) := x'(t)$, we will get a solution to the first order system IVP

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cancel{7}\,6 & \cancel{6}\,{-7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} .$$

②

$$\begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -6x - 7x' \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

b) Show without finding a formula for the solution $[x_1(t), x_2(t)]^T$ to the IVP in a, that if we define $x(t) := x_1(t)$, then $x(t)$ solves the original second order IVP.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ -6x_1 - 7x_2 \end{bmatrix}$$

$$\Rightarrow x_1'' = x_2' = -6x_1 - 7x_2$$
$$x_1'' = -6x_1 - 7x_1'$$
$$x_1'' + 7x_1' + 6x_1 = 0$$
$$x_1(0) = 1$$
$$x_1'(0) = x_2(0) = 4$$

~~c) Solve the second order DE IVP for $x(t)$ in order to deduce a solution to the first system order IVP. Use Chapter 3 methods.~~

$$B\vec{v} = \begin{bmatrix} col_1(B) & \vdots & col_2(B) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$$

$$= v_1 \, col_1(B) + v_2 \, col_2(B) = \vec{0}$$

d) Solve the first order system IVP in a using eigenvalue-eigenvector methods. Notice that the matrix "characteristic polynomial" is the same as the Chapter 3 "characteristic polynomial". Check that the $x_1(t)$ you find, i.e. the first component function of the system solution, is actually the $x(t)$ you found in part c.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix}$$

$$= \lambda(\lambda+7) + 6 = \lambda^2 + 7\lambda + 6$$

$$= (\lambda+6)(\lambda+1) = 0$$

$$\lambda = -1, -6.$$

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -10 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$$E_{\lambda=-1} \quad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -6 & -6 & 0 \end{array}\right] \qquad (A-\lambda I)\vec{v} = \vec{0}$$

$$\lambda = -1$$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$E_{\lambda=-6} \quad \left[\begin{array}{cc|c} 6 & 1 & 0 \\ -6 & -1 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}, \quad e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

e) Understand the phase portrait below, for the first order system IVP, in terms of the over-damped mass motion of the original second order differential equation. Also notice the influence of the eigenspaces in the solution trajectory!

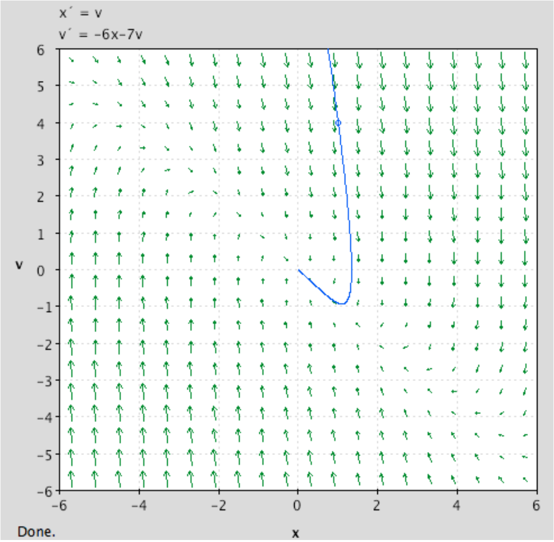
So, $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

soln to ①, is $x_1(t)$

$$x_1(t) = 2e^{-t} - e^{-6t}$$

$$① \begin{cases} x'' + 7x' + 6x = 0 \\ x(0) = 1 \\ x'(0) = 4 \end{cases}$$

$$p(r) = r^2 + 7r + 6$$
$$= (r+6)(r+1)$$



Math 2280-001
Week 9, March 6-10 5.1-5.3
Mon Mar 6
5.1-5.2 Linear systems of differential equations

Monday: Finish Friday's notes
Hw: the §4.1 material due <u>next</u> assignment
3/21
(you'll start w8.4
in class today, & I'll pull up pplane).

- Finish last Friday's notes to understand why any differential equation or system of differential equations can be converted into an equivalent (larger) system to a first order differential equations, and to overview the methods we will be using to solve first order systems of differential equations.

- Then proceed ...

Theorems for linear systems of differential equations:

<u>Theorem 1</u>   For the IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}})$ is continuous in the $t-$variable and differentiable in its $\underline{\boldsymbol{x}}$ variable, then there exists a unique solution to the IVP, at least <u>on some (possibly short) time interval</u> $t_0 - \delta < t < t_0 + \delta$.

<u>Theorem 2</u>   For the special case of the first order <u>linear system of differential equations</u> IVP

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, <u>on the entire interval</u>.

<u>"new" today:</u>

<u>Theorem 3)</u> Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness <u>Theorem 2</u> as fact.)

<u>3.1)</u> For vector functions $\underline{\boldsymbol{x}}(t)$ differentiable on an interval, the operator

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t)$$

is linear, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{z}}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{\boldsymbol{z}}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t)) .$$

check!