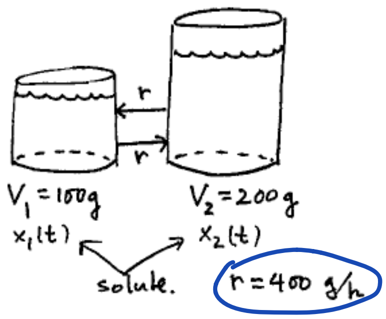
Quiz 9-9:15
9:15-9:25 for last HW problem

Math 2280-001
Fri Mar 3

4.1 Systems of differential equations - to model multi-component systems via compartmental analysis

http://en.wikipedia.org/wiki/Multi-compartment_model
and to understand higher order differential equations.

Here's a relatively simple 2-tank problem to illustrate the ideas:



Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$$x_1'(t) = r_i c_i - r_o c_o$$
$$= 400 \cdot \frac{x_2}{200} - 400 \frac{x_1}{100} = 2x_2 - 4x_1 = -4x_1 + 2x_2$$
$$\left(\frac{g}{h}\right)\left(\frac{\text{mass}}{g}\right)$$
$$x_2'(t) = r_i c_i - r_o c_o$$
$$= 400 \frac{x_1}{100} - 400 \frac{x_2}{200} = 4x_1 - 2x_2$$

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

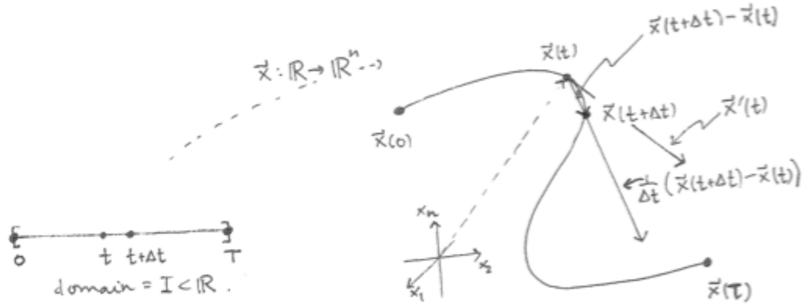
The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

$$\boxed{\begin{aligned} \underline{\mathbf{x}}'(t) &= \underline{\mathbf{F}}(t, \underline{\mathbf{x}}(t)) \\ \underline{\mathbf{x}}(t_0) &= \underline{\mathbf{x}}_0 \end{aligned}}$$

- We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

Why we expect IVP's for first order systems of DE's to have unique solutions $\underline{\mathbf{x}}(t)$:

- From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{\mathbf{x}}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{\mathbf{x}}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:



Analytically, the reason that the vector of derivatives $\underline{\mathbf{x}}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\underline{\mathbf{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ \vdots \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}\right)$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0$), and you know your "velocity" vector (depending on time and current location) $\Rightarrow$ you expect a unique solution! (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! And this is what numerical solvers do.)

These are vector analogs of the theorems we discussed in Chapter 1 for first order scalar differential equations. The first one should make intuitive sense, based on the reasoning of the previous page.

Theorem 1 For the IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}})$ is continuous in the t-variable and differentiable in its $\underline{\boldsymbol{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

$\vec{x}'(t) - A(t)\vec{x}(t) = \vec{f}(t)$

[chptr 1 $x' + P(t)x = Q(t)$]

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. For example, with time-step h the Euler loop would increment as follows:

$$t_{j+1} = t_j + h$$
$$\underline{\boldsymbol{x}}_{j+1} = \underline{\boldsymbol{x}}_j + h\,\underline{\boldsymbol{F}}(t_j, \underline{\boldsymbol{x}}_j).$$

Remark: These theorems are the true explanation for why the n^{th}-order linear DE IVPs in Chapter 3 always have solutions - We will see that each $n^{th} - order$ linear DE IVP is actually equivalent to an IVP for a first order system of n linear DE's. (The converse is not true.) In fact, when software finds numerical approximations for solutions to higher order (linear or non-linear) DE IVPs that can't be found by the techniques of Chapter 3 or other mathematical formulas, it converts these IVPs to the equivalent first order system IVPs, and uses algorithms like Euler and Runge-Kutta to approximate the solutions.

Exercise 2) Return to the page 1 tank example

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4x_1 + 2x_2 \\ 4x_1 - 2x_2 \end{bmatrix} = \begin{matrix} (4x_1 - 2x_2) \\ 2(2x_1 - x_2) \end{matrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

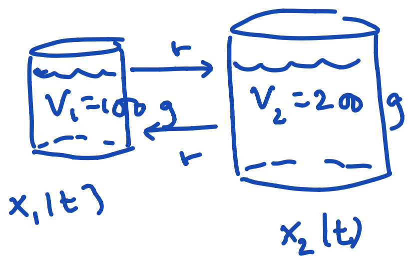
$$x_1(0) = 9$$

$$x_2(0) = 0$$

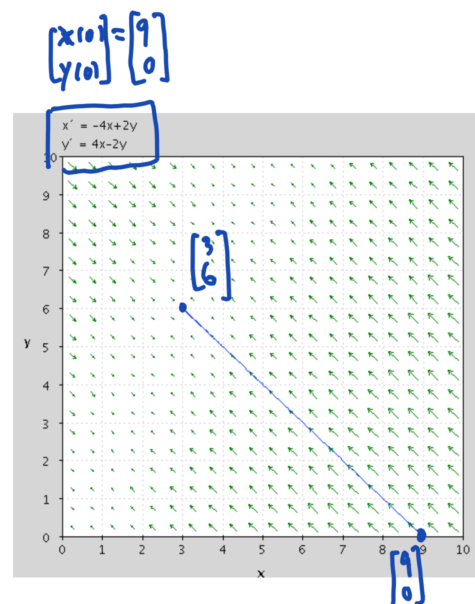
2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

2b) What are the apparent limiting solute amounts in each tank? $\rightarrow$ $x_1(t) \to 3$, $x_2(t) \to 6$

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?



$r = 400 \frac{g}{h}$



$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

First order systems of differential equations ~~of the form~~

$$\boxed{\underline{\mathbf{x}}'(t) = A(t)\,\underline{\mathbf{x}}}$$

are called linear homogeneous systems of DE's. (Think of rewriting the system as

$$\boxed{\underline{\mathbf{x}}'(t) - A(t)\,\underline{\mathbf{x}} = \underline{\mathbf{0}}}$$

in analogy with how we wrote linear scalar differential equations.) Then the inhomogeneous system of first order DE's would be written as

$$\boxed{\underline{\mathbf{x}}'(t) - A(t)\,\underline{\mathbf{x}} = \underline{\mathbf{f}}(t)}$$

or

$$\underline{\mathbf{x}}'(t) = A(t)\,\underline{\mathbf{x}} + \underline{\mathbf{f}}(t)$$

Exercise 3a) Show the space of solutions $\underline{\mathbf{x}}(t)$ to the homogeneous system of DE's

$$\underline{\mathbf{x}}'(t) - A(t)\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

is a subspace, i.e. linear combinations of solutions are solutions.

$L(\vec{x}(t)) = x'(t) - A(t)\vec{x}(t)$

$V = \{\vec{x}(t) : I \to \mathbb{R}^n,$ s.t. $\vec{x}(t)$ is continuous, $\vec{x}'(t)$ is continuous$\}$

Check L is linear

$L : V \to W$

$W = \{\vec{y}(t) : I \to \mathbb{R}^n$ s.t. $\vec{y}(t)$ is cont$\}$

(i) $L(\vec{x}(t) + \vec{z}(t)) = L(\vec{x}(t)) + L(\vec{z}(t))$

$\| \quad (x' + z') - A(x+z)$

$\| \quad x' + z' - Ax - Az$

$= (x' - Ax) + (z' - Az)$

$= L(x) + L(z)$

(ii) $L(c\vec{x}(t))$

$= (c\vec{x})' - Ac\vec{x}$

$= c(\vec{x}' - A\vec{x})$

$= cL(\vec{x})$

3b) In the special case that A is a constant matrix ("constant coefficients"), look for a basis of solutions of the form

$$\underline{\mathbf{x}}(t) = e^{\lambda t}\underline{\mathbf{v}},$$

where $\underline{\mathbf{v}}$ is a constant vector. Hint: In order for such an $\underline{\mathbf{x}}(t)$ to solve the DE it must be true that

$$\underline{\mathbf{x}}'(t) = \lambda e^{\lambda t}\underline{\mathbf{v}}$$

and

$$A\,\underline{\mathbf{x}}(t) = A e^{\lambda t}\underline{\mathbf{v}} = e^{\lambda t}A\,\underline{\mathbf{v}}$$

must agree. *These functions of t will agree if and only* $\lambda\,\underline{\mathbf{v}} = A\,\underline{\mathbf{v}}$. So, it's time to recall eigenvalues and eigenvectors! (Math 2270).

$\lambda\vec{v} = A\vec{v}$

get soltns!

$A\vec{v} - \lambda\vec{v} = \vec{0}$

$A\vec{v} - \lambda I\vec{v} = \vec{0}$

$\boxed{(A - \lambda I)\vec{v} = \vec{0}}$

3c) Solve the initial value problem of Exercise 2!! Compare your solution $\underline{\mathbf{x}}(t)$ to the parametric curve on the previous page.

$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}. \end{cases}$$

1

eigenvalues & eigenvectors

$|A - \lambda I| = 0$

$$\begin{vmatrix} -4-\lambda & 2 \\ 4 & -2-\lambda \end{vmatrix} = (\lambda+4)(\lambda+2) - 8$$

$= \lambda^2 + 6\lambda + \cancel{8} - \cancel{8}$

$= \lambda(\lambda+6) = 0$

$\lambda = 0, -6.$

$E_{\lambda=0}$ $(A - 0I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$R_1/2$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ soltn $e^{\lambda t}\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $(e^{0t}\vec{v})$

$E_{\lambda=-6}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

soln $e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

IVP: $\begin{bmatrix} 9 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -9 \\ -18 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

long way.

$$\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_1 = \frac{1}{2}t$, $v_2 = t$, $\vec{v} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

@ $t=0$: $\begin{bmatrix} 9 \\ 0 \end{bmatrix}$ ✓

Exercise 4) Lessons learned from Exercise 3:

a) What condition on the constant matrix $A_{n\times n}$ will allow you to solve every initial value problem

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \in \mathbb{R}^n$$

using the method in Exercise 3 ? Hint: Math 2270 discussions (If that condition fails there are other ways to find the unique solutions.)

short way.

$A\vec{v} = v_1\,col_1(A) + v_2\,col_2(A) + \dots + v_n\,col_n(A)$

$1\cdot col_1(A) + 2\,col_2(A) = \vec{0}$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{u}(t) = e^{0t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}(t) = e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

are a basis for the sol'n space to $\vec{x}' = A\vec{x}$ is #3.

(1) span: Can solve any IVP with $\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$

at $t=0$: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

and each IVP only has one sol'n (any sol'n has initial vector)

so every sol'n. is a linear combo.

b) What is the dimension of the solution space to the first order system

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

when $\underline{\boldsymbol{x}}_0 \in \mathbb{R}^n$ and $A = A_{n\times n}$?

(2) independence.

$c_1\vec{u}(t) + c_2\vec{v}(t) \equiv 0$

@ $t=0$. $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = 0$.

(4) $\vec{x}'(t) = A\vec{x}$ $\quad A_{n\times n}$ constant matrix

If there's a basis of $\mathbb{R}^n$, $\mathbb{C}^n$, $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ made out of eigenvectors of A

$A\vec{v}_j = \lambda_j\vec{v}_j$.

then $\vec{x}_H(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + \dots + c_ne^{\lambda_n t}\vec{v}_n$

to solve $\vec{x}(0) = \vec{b}$

$\vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

$$\vec{b} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Converting higher order DE's or systems of DE's to equivalent first order systems of DEs:

It is always the case that the natural initial value problems for single differential equations or systems of differential equations are equivalent to initial value problems for larger systems of first order differential equations. The following discussion will illustrate this equivalence.

For example, consider this second order underdamped IVP for $x(t)$:

$$x'' + 2x' + 5x = 0$$
$$x(0) = 4$$
$$x'(0) = -4 \ .$$

1

Exercise 5)

5a) Suppose that $x(t)$ solves the IVP above. Define $x_1(t) := x(t)$ and $x_2(t) := x'(t)$. Show that $[x_1, x_2]^T$ solves the first order system initial value problem

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}.$$

2

$$\frac{d}{dt}\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -5x-2x' \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

2

5b) Conversely, show that if $[x_1, x_2]^T$ solves the first order system in 5a, then its first component function $x(t) := x_1(t)$ solves the original second order differential equation initial value problem. Thus, there is an equivalence between the original second order DE initial value problem, and the corresponding IVP for the related system of first order DE's.

$$x_1' = x_2 \longrightarrow \frac{d}{dt}: \ x_1'' = x_2' = -5x_1 - 2x_2$$
$$x_2' = -5x_1 - 2x_2$$
$$x_1'' = -5x_1 - 2x_1'$$
$$x_1'' + 2x_1' - 5x_1 = 0$$
$$x_1(0) = 4$$
since $x_2 = x_1'$, $x_1'(0) = -4$

5c) Solve the second order IVP in order to deduce a solution to the first order IVP in 5a. (Use Chapter 3 techniques.)
5d) How does the Chapter 3 "characteristic polynomial" in the second order differential equation compare with (Math 2270) eigenvalue "characteristic polynomial" for the matrix in the first order system 5a? hmmm. What if you used complex eigenvalues and eigenvectors to solve the first order system IVP. Could you recover the solution $x(t)$ to the original second order DE IVP?
5d) Is your analytic solution $[x_1(t), x_2(t)] = [x(t), v(t)]$ in 5a consistent with the parametric curve shown below, in a "pplane" screenshot? The picture is called a "phase portrait" for position and velocity.

5c) $x'' + 2x' + 5x = 0$

$x(0) = 4$

$x'(0) = -4$

$\boxed{p(r) = r^2 + 2r + 5} = (r+1)^2 + 4 = 0$

$(r+1)^2 = -4$

$r + 1 = \pm 2i$

$r = -1 \pm 2i$

$x(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

$x(0) = 4 = c_1$

$x'(0) = -4 = -c_1 + 2c_2$

$\Rightarrow c_1 = 4,\ c_2 = 0.$

$\boxed{x(t) = 4e^{-t}\cos 2t}$ (1)

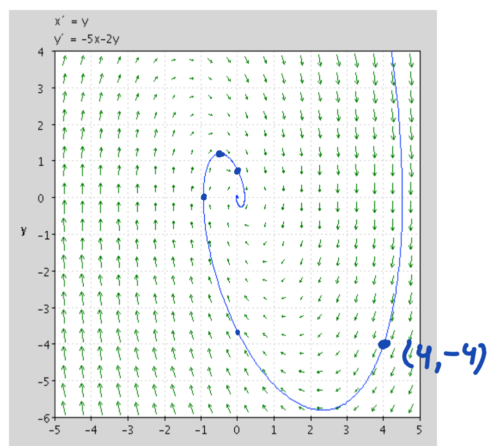
$x'(t) = c_1(-e^{-t}\cos 2t - 2e^{-t}\sin 2t) + c_2(-e^{-t}\sin 2t + 2e^{-t}\cos 2t)$

So sol'n to 1st order system (2) is

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} 4e^{-t}\cos 2t \\ -4e^{-t}\cos 2t - 8e^{-t}\sin 2t \end{bmatrix}$$

$$= \underbrace{4e^{-t}\underbrace{\begin{bmatrix} 1 & 0 \\ -1 & +2 \end{bmatrix}\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}}_{\text{ellipse}}}_{\text{spiral converging exponentially to } \vec{0}}$$

clockwise circle



5d): $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

look for a basis of sol'ns $e^{\lambda t}\vec{v}$

$A\vec{v} = \lambda\vec{v}.$

$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = \lambda(\lambda+2) + 5$

$\boxed{= \lambda^2 + 2\lambda + 5} = 0$

$|\lambda I - A|$

$\boxed{\lambda = -1 \pm 2i}$

$e^{\lambda t}\vec{v}$?? & $e^{\bar{\lambda} t}\bar{\vec{v}}$

Exercise 6)

consider this second order ~~underdamped~~ overdamped IVP for $x(t)$:

$$\cancel{x'' + 6x' + 7x = 0} \qquad x'' + 7x' + 6x = 0$$

$$x(0) = 1$$
$$x'(0) = 4 \ .$$

1

a) Show without finding a formula for the solution $x(t)$, that whatever the function is, then by defining $x_1(t) := x(t)$ and $x_2(t) := x'(t)$, we will get a solution to the first order system IVP

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cancel{7}\,6 & \cancel{6}\,{-7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

2

$$\begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ x' \end{bmatrix}' = \begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} x' \\ -6x - 7x' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

b) Show without finding a formula for the solution $[x_1(t), x_2(t)]^T$ to the IVP in a, that if we define $x(t) := x_1(t)$, then $x(t)$ solves the original second order IVP.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ -6x_1 - 7x_2 \end{bmatrix}$$

$$\Rightarrow x_1'' = x_2' = -6x_1 - 7x_2$$
$$x_1'' = -6x_1 - 7x_1'$$
$$x_1'' + 7x_1' + 6x_1 = 0$$
$$x_1(0) = 1$$
$$x_1'(0) = x_2(0) = 4$$

~~c) Solve the second order DE IVP for $x(t)$ in order to deduce a solution to the first system order IVP. Use Chapter 3 methods.~~

$$B\vec{v} = \left[\mathrm{col}_1(B) \,\vdots\, \mathrm{col}_2(B)\right]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$$

$$= v_1\,\mathrm{col}_1(B) + v_2\,\mathrm{col}_2(B) = \vec{0}$$

d) Solve the first order system IVP in a using eigenvalue-eigenvector methods. Notice that the matrix "characteristic polynomial" is the same as the Chapter 3 "characteristic polynomial". Check that the $x_1(t)$ you find, i.e. the first component function of the system solution, is actually the $x(t)$ you found in part c.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix}$$

$$= \lambda(\lambda+7) + 6 = \lambda^2 + 7\lambda + 6$$

$$= (\lambda+6)(\lambda+1) = 0$$

$$\lambda = -1, -6.$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

$E_{\lambda=-1}$ $\quad \lambda = -1$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -6 & -6 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \leftarrow$$

$E_{\lambda=-6}$

$$\left[\begin{array}{cc|c} 6 & 1 & 0 \\ -6 & -1 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}, \quad e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix} \leftarrow$$

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -6 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-5}\begin{bmatrix} -6 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -10 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

e) Understand the phase portrait below, for the first order system IVP, in terms of the over-damped mass motion of the original second order differential equation. Also notice the influence of the eigenspaces in the solution trajectory!

$$x_1' = x_2$$
$$x_2' = -6x_1 - 7x_2$$

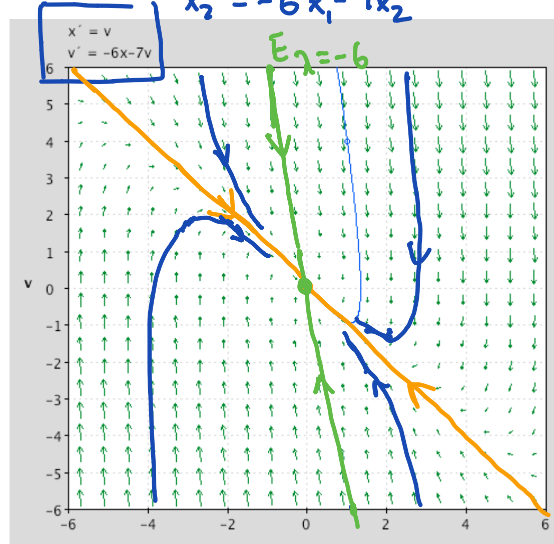
So,
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

soln to (1), is $x_1(t)$

$$x_1(t) = 2e^{-t} - e^{-6t}$$

$$(1)\ \begin{cases} x'' + 7x' + 6x = 0 \\ x(0) = 1 \\ x'(0) = 4 \end{cases}$$

$$p(r) = r^2 + 7r + 6$$
$$= (r+6)(r+1)$$



Exercise 7) Consider an n^{th} order differential equation for a function $x(t)$:

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \ldots x^{(n-1)}(t)\right)$$

and the IVP

$$\begin{gathered} x^{(n)} = f\left(t, x, x', \ldots x^{(n-1)}\right) \\ x(t_0) = b_0 \\ x'(t_0) = b_1 \\ x''(t_0) = b_2 \\ \vdots \\ x^{(n-1)}(t_0) = b_{n-1}. \end{gathered}$$

can skip

a) Show that if $x(t)$ solves the IVP above, and if we define functions $x_1(t), x_2(t), \ldots x_n(t)$ by

$$x_1 := x,\ x_2 := x', x_3 := x'', \ldots x_n := x^{(n-1)}$$

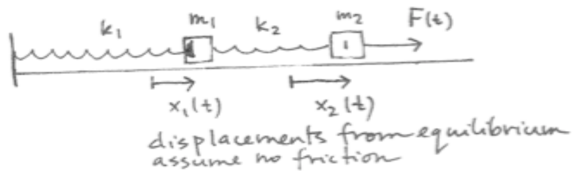
Then $[x_1, x_2, \ldots, x_n]$ solve the first order system IVP

$$\begin{gathered} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = f\left(t, x_1, x_2, \ldots x_{n-1}\right) \\ x_1(t_0) = b_0 \\ x_2(t_0) = b_1 \\ x_3(t_0) = b_2 \\ \vdots \\ x_n(t_0) = b_{n-1} \end{gathered}$$

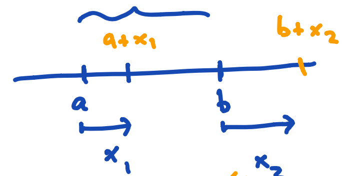
b) Show that if $[x_1(t), x_2(t), \ldots x_n(t)]$ is a solution to the IVP in a, then the first function $x_1(t)$ solves the original IVP for the n^{th} order differential equtation.

Higher order systems of DE's are also equivalent to first order systems, as illustrated in the next example.

Consider this configuration of two coupled masses and springs:



equil dist $= b-a$.



new dist $= (b+x_2)-(a+x_1)$
$= (b-a)+(x_2-x_1)$

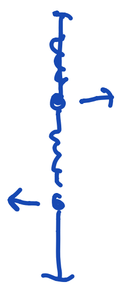
additional separation

Exercise 8) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case?

$$m_1 x_1'' = \text{net forces} = F_{\text{spring } 1} + F_{\text{spring } 2}$$

$$\begin{cases} m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' = -k_2(x_2 - x_1) + F(t) \end{cases}$$

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2) x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - k_2 x_2 + F(t) \end{cases}$$



OR
(see hw)

Exercise 9) Consider the IVP from Exercise 6, with the special values
$m_1 = 2, m_2 = 1; k_1 = 4, k_2 = 2; F(t) = 40\sin(3t)$:

$$m_1 x_1'' = -(\underbrace{k_1+k_2}_{6})x_1 + \underbrace{k_2}_{2} x_2$$
$$m_2 x_2'' = k_2 x_1 - k_2 x_2 + F(t)$$

2 ; 1 ; 2 ; 2 ; 40 sin 3t

1
$$x_1'' = -3x_1 + x_2$$
$$x_2'' = 2x_1 - 2x_2 + 40\sin(3t)$$
$$x_1(0) = b_1, x_1'(0) = b_2$$
$$x_2(0) = c_1, x_2'(0) = c_2 .$$

9a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define
$$v_1(t) := x_1'(t)$$
$$v_2(t) := x_2'(t)$$
then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

2
$$x_1' = v_1$$
$$x_2' = v_2$$
$$x_1'' = v_1' = -3x_1 + x_2$$
$$x_2'' = v_2' = 2x_1 - 2x_2 + 40\sin(3t)$$
$$x_1(0) = b_1$$
$$v_1(0) = b_2$$
$$x_2(0) = c_1$$
$$v_2(0) = c_2 .$$

soln to 1
$x_1(t), x_2(t)$

yields soln to 2
$x_1(t)$
$v_1(t)$
$x_2(t)$
$v_2(t)$

9b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.

soln to 2 yields soln to 1

$x_1' = v_1$.

$x_1'' = v_1' = -3x_1 + x_2$

Math 2280-001
Week 9, March 6-10 5.1-5.3
Mon Mar 6
5.1-5.2 Linear systems of differential equations

Monday: Finish Friday's notes
Hw: the §4.1 material due next assignment 3/21
(you'll start w8.4 in class today, & I'll pull up pplane).

- Finish last Friday's notes to understand why any differential equation or system of differential equations can be converted into an equivalent (larger) system to a first order differential equations, and to overview the methods we will be using to solve first order systems of differential equations.

- Then proceed ...

Theorems for linear systems of differential equations:

Theorem 1 For the IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}})$ is continuous in the $t-$variable and differentiable in its $\underline{\boldsymbol{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

"new" today:

Theorem 3) Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{\boldsymbol{x}}(t)$ differentiable on an interval, the operator

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t)$$

is linear, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{z}}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{\boldsymbol{z}}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t)).$$

check!

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{f}}(t)$$

$\forall\ t \in I$ is

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_p(t) + \underline{\boldsymbol{x}}_H(t)$$

where $\underline{\boldsymbol{x}}_p(t)$ is any single particular solution and $\underline{\boldsymbol{x}}_H(t)$ is the general solution to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}}$$

We frequently write the homogeneous linear system of DE's as

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t)\ .$$

3.3) For $A(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the $t-$interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

is n-dimensional. Here's why:

- Let $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$ defined by

$$\left[W\left(\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \ldots, \underline{\boldsymbol{X}}_n\right)\right]\left(t_0\right) := \left[\underline{\boldsymbol{X}}_1\left(t_0\right)\middle|\underline{\boldsymbol{X}}_2\left(t_0\right)\middle| \cdots \middle|\underline{\boldsymbol{X}}_n\left(t_0\right)\right]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible Wronskian matrix.)

- Then for any $\underline{\boldsymbol{b}} \in \mathbb{R}^n$ the IVP

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}\left(t_0\right) = \underline{\boldsymbol{b}}$$

has solution $\underline{\boldsymbol{x}}(t) = c_1\underline{\boldsymbol{X}}_1(t) + c_2\underline{\boldsymbol{X}}_2(t) + ... + c_n\underline{\boldsymbol{X}}_n(t)$ where the linear combination coefficients comprise the solution vector to the Wronskian matrix equation

$$\left[\begin{array}{c|c|c|c} \underline{\boldsymbol{X}}_1\left(t_0\right) & \underline{\boldsymbol{X}}_2\left(t_0\right) & \cdots & \underline{\boldsymbol{X}}_n\left(t_0\right) \end{array}\right]\begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix}.$$

$[W]\vec{c} = \vec{b}$

$\vec{c} = [W]^{-1}\vec{b}$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$, and since each IVP has only one solution, $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$) is the one with $\underline{\boldsymbol{c}} = \underline{\boldsymbol{0}}$. Thus $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and their number n is the dimension of the solution space.

y_H : $p(r) = r^3 + a_2 r^2 + a_1 r + a_0$

$\vec{Z}_H$: $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & -a_2-\lambda \end{vmatrix} = -a_0(1) + a_1(-\lambda) + (-a_2-\lambda)(\lambda^2) = -[\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0]$

<u>Remark:</u> If the first order system arises by converting an n^{th} order linear differential equation for $y(x)$ into a linear system of n first order differential equations, then solutions $y(x)$ to the DE correspond to vector solutions $\left[y(x), y'(x), \ldots, y^{(n-1)}(x)\right]$ for the first order systems, so the Chapter 3 Wronskians correspond exactly to the Chapter 5 Wronskians.

<u>Exercise 1)</u> Check this Remark. $n = 3$.

$y''' + a_2 y'' + a_1 y' + a_0 y = f$

y_H : $f = 0$.

y_1, y_2, y_3

$W(y_1, y_2, y_3) = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$

$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} y' \\ y'' \\ -a_0 y - a_1 y' - a_2 y'' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cancel{f} \end{bmatrix} \to 0$ for y_H

$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$

$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}' = \begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

<u>Theorem 4)</u> The eigenvalue-eigenvector method for a solution space basis to the homogeneous system (as discussed informally in last week's notes and examples): For the system

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

with $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$, $A_{n \times n}$, if the matrix A is diagonalizable (i.e. there exists a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_n\}$ of $\mathbb{R}^n$ made out of eigenvectors of A, i.e. $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ for each $j = 1, 2, \ldots, n$), then the functions

$$e^{\lambda_j t}\,\underline{\boldsymbol{v}}_j\,, \quad j = 1, 2, \ldots n$$

are a basis for the (homogeneous) solution space, i.e. each solution is of the form

$$\underline{\boldsymbol{x}}_H(t) = c_1 e^{\lambda_1 t}\,\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\,\underline{\boldsymbol{v}}_2 + \ldots + c_n e^{\lambda_n t}\,\underline{\boldsymbol{v}}_n\,.$$

<u>proof</u>: check the Wronskian matrix at $t = 0$, its the matrix that has the eigenvectors in its columns, and is invertible because they're a basis for $\mathbb{R}^n$.

sol'tns $\vec{Z}_1(t) = \begin{bmatrix} y_1 \\ y_1' \\ y_1'' \end{bmatrix}$, $\vec{Z}_2 = \begin{bmatrix} y_2 \\ y_2' \\ y_2'' \end{bmatrix}$, $\vec{Z}_3 = \begin{bmatrix} y_3 \\ y_3' \\ y_3'' \end{bmatrix}$

$W(Z_1, Z_2, Z_3) = \left[\vec{Z}_1 \middle| \vec{Z}_2 \middle| \vec{Z}_3\right]$

• We will continue using the eigenvalue-eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

that we discussed last week.

So far we've not considered the possibility of complex eigenvalues and eigenvectors. Linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc. Then the model space is $\mathbb{C}^n$ rather than $\mathbb{R}^n$.

Definition: $\underline{\boldsymbol{v}} \in \mathbb{C}^n$ ($\underline{\boldsymbol{v}} \neq \underline{\mathbf{0}}$) is a complex eigenvector of the matrix A, with eigenvalue $\lambda \in \mathbb{C}$ if $A\,\underline{\boldsymbol{v}} = \boldsymbol{\lambda}\,\underline{\boldsymbol{v}}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work an example. We can also refer to the general discussion on the following pages, at appropriate stages.

Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

$$G' = -.1G - .4H$$
$$H' = .1G - .1H$$

Exercise 1a) Understand why the signs of the matrix entries are reasonable.

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

1b) The first step is to get the eigendata of the matrix. Do this, and compare with the Maple check on the next page.

$$\begin{vmatrix} -.1-\lambda & -.4 \\ .1 & -.1-\lambda \end{vmatrix} = (\lambda+.1)^2 + .04 = 0$$
$$(\lambda+.1)^2 = -.04$$
$$\lambda + .1 = \pm .2i$$
$$\lambda = -.1 \pm .2i$$

$\lambda = -.1 \pm .2i$

$[A - \lambda I]\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -.2i & -.4 & 0 \\ .1 & -.2i & 0 \end{array}\right]$$

$$\begin{array}{l} 10R_2 \\ 5R_1 \end{array} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ -i & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ iR_1 + R_2 \end{array} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$\vec{z}(t) = e^{\lambda t}\vec{v}$$
$$= e^{(-.1+\underline{.2i})t} \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$
$$= e^{-.1t}\left(\cos(.2t) + i\sin(.2t)\right) \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$
$$= e^{-.1t} \begin{bmatrix} -2\sin(.2t) \\ \cos(.2t) \end{bmatrix} + i e^{-.1t} \begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix}$$

$$\vec{Z}(t) = \vec{X}(t) + i\vec{Y}(t)$$
$$\overline{\vec{Z}}(t) = \vec{X}(t) - i\vec{Y}(t)$$

> with(LinearAlgebra) :

> $A := \begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$;

Eigenvectors(A);

$$A := \begin{bmatrix} -\frac{1}{10} & -\frac{2}{5} \\ \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{10} + \frac{1}{5} I \\ -\frac{1}{10} - \frac{1}{5} I \end{bmatrix}, \begin{bmatrix} 2 I & -2 I \\ 1 & 1 \end{bmatrix} \quad (1)$$

Notice that Maple writes a capital $I = \sqrt{-1}$.

1c) Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

$$\vec{Z}(t) = \vec{X}(t) + i\vec{Y}(t).$$

$$\vec{Z}'(t) = A\vec{Z}$$

$$\vec{X}'(t) + i\vec{Y}'(t) = A(\vec{X} + i\vec{Y})$$

$$\vec{X}' + i\vec{Y}' = A\vec{X} + iA\vec{Y}$$

$$\Rightarrow \vec{X}' = A\vec{X}$$

$$\vec{Y} = A\vec{Y}$$

So $\left\{ e^{-.1t}\begin{bmatrix} -2\sin(.2t) \\ \cos(.2t) \end{bmatrix}, e^{-.1t}\begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix} \right\}$ is a basis of real fcns!

1d) Solve the initial value problem.

$$\begin{bmatrix} G' \\ H' \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}\begin{bmatrix} G \\ H \end{bmatrix} \qquad \begin{bmatrix} G \\ H \end{bmatrix} = c_1 e^{-.1t}\begin{bmatrix} -2\sin(.2t) \\ \cos(.2t) \end{bmatrix} + c_2 e^{-.1t}\begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix}$$

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \qquad @\, t=0: \begin{bmatrix} 100 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

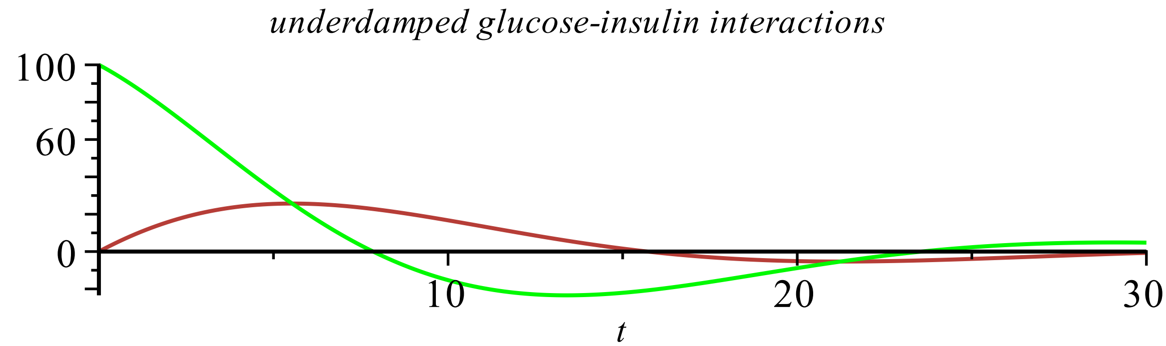
$$\begin{bmatrix} G \\ H \end{bmatrix} = 50 e^{-.1t}\begin{bmatrix} 2\cos(.2t) \\ \sin(.2t) \end{bmatrix} \qquad \Rightarrow c_1 = 0,\ c_2 = 50$$

Here are some pictures to help understand what the model is predicting ... you could also construct these graphs using pplane.

(1) Plots of glucose vs. insulin, at time t hours later:

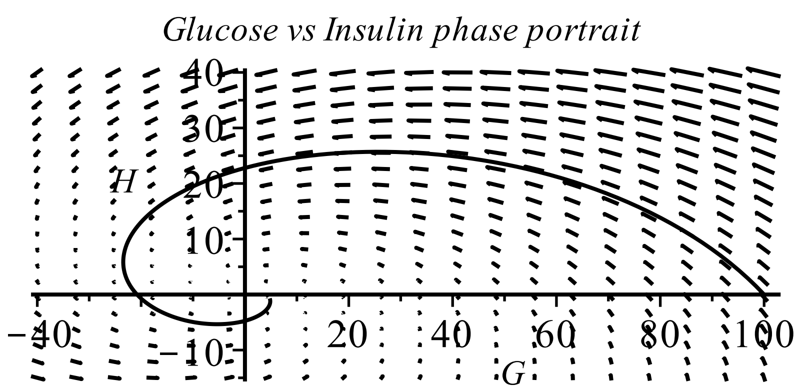
```
> with(plots) :
```

```
> G := t→100·exp(−.1·t)·cos(.2·t) :
  H := t→50·exp(−.1·t)·sin(.2·t) :
  plot1 := plot(G(t), t = 0 ..30, color = green) :
  plot2 := plot(H(t), t = 0 ..30, color = brown) :
  display({plot1, plot2}, title = `underdamped glucose-insulin interactions`);
```



2) A phase portrait of the glucose-insulin system:

```
> pict1 := fieldplot([−.1·G − .4·H, .1·G − .1·H], G = −40 ..100, H = −15 ..40) :
  soltn := plot([G(t), H(t), t = 0 ..30], color = black) :
  display({pict1, soltn}, title = `Glucose vs Insulin phase portrait`);
```



Solutions to homogeneous linear systems of DE's when matrix has complex eigenvalues:

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

Let A be a real number matrix. Let

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \;\in \mathbb{C}^n$$

satisfy $A\,\underline{\boldsymbol{v}} = \boldsymbol{\lambda}\,\underline{\boldsymbol{v}}$, with $a, b \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}^n$.

- Then $\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ is a complex solution to

$$\underline{\boldsymbol{x}}'(t) = A\,\boldsymbol{x}$$

because $\underline{\boldsymbol{z}}'(t) = \lambda e^{\lambda t}\,\underline{\boldsymbol{v}}$ and this is equal to $A\,\underline{\boldsymbol{z}} = \boldsymbol{A}\,e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t}A\,\underline{\boldsymbol{v}}$.

- But if we write $\underline{\boldsymbol{z}}(t)$ in terms of its real and imaginary parts,

$$\underline{\boldsymbol{z}}(t) = \underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t)$$

then the equality

$$\underline{\boldsymbol{z}}'(t) = A\,\boldsymbol{z}$$
$$\Rightarrow \underline{\boldsymbol{x}}'(t) + i\,\underline{\boldsymbol{y}}'(t) = A(\underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t)) = A\,\underline{\boldsymbol{x}}(t) + i\,A\,\underline{\boldsymbol{y}}(t)\,.$$

Equating the real and imaginary parts on each side yields

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}(t)$$
$$\underline{\boldsymbol{y}}'(t) = A\,\underline{\boldsymbol{y}}(t)$$

i.e. the real and imaginary parts of the complex solution are each real solutions.

- If $A\,(\underline{\alpha} + i\,\underline{\boldsymbol{\beta}}) = (a + b\,i)(\underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}})$ then it is straightforward to check that $A\,(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}) = (a - b\,i)(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}})$. Thus the complex conjugate eigenvalue yields the complex conjugate eigenvector. The corresponding complex solution to the system of DEs

$$e^{(a - i\,b)t}(\underline{\alpha} - i\,\underline{\boldsymbol{\beta}}) = \underline{\boldsymbol{x}}(t) - i\,\underline{\boldsymbol{y}}(t)$$

so yields the same two real solutions (except with a sign change on the second one).

- More details of what the real soluitions look like:

$$\lambda = a + b\,i \in \mathbb{C}$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{\alpha}} + i\,\underline{\boldsymbol{\beta}} \;\in \mathbb{C}^n$$

$$\Rightarrow \mathrm{e}^{\lambda t}\underline{\boldsymbol{v}} = \mathrm{e}^{a\,t}(\cos(b\,t) + i\sin(b\,t))\cdot(\underline{\boldsymbol{\alpha}} + \underline{\boldsymbol{\beta}}\,i) = \underline{\boldsymbol{x}}(t) + i\,\underline{\boldsymbol{y}}(t).$$

So the real solutions are

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{a\,t}(\cos(bt)\underline{\boldsymbol{\alpha}} - \sin(bt)\underline{\boldsymbol{\beta}})$$
$$\underline{\boldsymbol{y}}(t) = \mathrm{e}^{a\,t}(\cos(bt)\underline{\boldsymbol{\beta}} + \sin(bt)\underline{\alpha})$$

- The Glucose-insulin example is linearized, and is vastly simplified. But mathematicians, doctors, bioengineers, pharmacists, are very interested in (especially more realistic) problems like these. Prof. Fred Adler and a recent graduate student Chris Remien in the Math Department, and collaborating with the University Hospital recently modeled liver poisoning by acetominophen (brand name Tyleonol), by studying a non-linear system of 8 first order differential equations. They came up with a state of the art and very useful diagnostic test:

http://unews.utah.edu/news_releases/math-can-save-tylenol-overdose-patients-2/

Here's a link to their published paper. For fun, I copied and pasted the non-linear system of first order differential equations from a preprint of their paper, below:

http://onlinelibrary.wiley.com/doi/10.1002/hep.25656/full

http://www.math.utah.edu/~korevaar/2250spring12/adler-remien-preprint.pdf

pted Prepr

Acetaminophen → APAP $$\frac{dA}{dt} = -\frac{\alpha}{H_{max}}AH - \delta_a A$$

liver toxin → NAPQI $$\frac{dN}{dt} = \frac{qp\alpha}{H_{max}}A - \gamma NG$$

Glutathione removes toxin → GSH $$\frac{dG}{dt} = \kappa - \gamma NG - \delta_g G$$

Functional Hepatocytes $$\frac{dH}{dt} = rH\left(1 - \frac{H+Z}{H_{max}}\right) - \eta NH$$

Damaged Hepatocytes $$\frac{dZ}{dt} = \eta NH - \delta_z Z$$

liver enzymes → AST $$\frac{dS}{dt} = \frac{d_z\beta_s}{\theta H_{max}}Z - \delta_s(S - S_{min})$$

ALT $$\frac{dL}{dt} = \frac{d_z\beta_l}{\theta H_{max}}Z - \delta_l(L - L_{min})$$

Clotting Factor $$\frac{dF}{dt} = \beta_f\left(\frac{H}{H_{max}} - F\right)$$

measure these

8 "INR" measure of liver damage w.r.t. clotting factors at admission

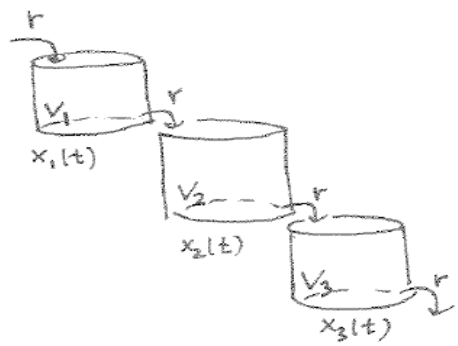
to recover info about

1) size of overdose

2) how long before admission

Math 2280-001
Wed Mar 8
5.2 The eigenvalue-eigenvector for solving homogeneous linear systems of DE's, continued.

Example 1) consider the three component input-output model below:



Exercise 1a) Derive the first order system for the tank cascade above.

1b) In case the tank volumes (in gallons) are $V_1 = 20$, $V_2 = 40$, $V_3 = 50$, the flow rate $r = 10\ \frac{gal}{\text{min}}$, and pure water (with no solute) is flowing into the first (top) tank, show that your system in (a) can be written as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

(This system is actually worked out in the text, page 287-288...but we'll modify the IVP, and then consider a second case as well.)

1c) Here's the eigenvector data for the matrix in b. You may want to check or derive parts of it by hand, especially if you're still not expert at finding eigenvalues and eigenvectors. I entered the matrix entries as rational numbers rather than decimals, because otherwise Maple would have given (confusing) floating point answers. Use the eigendata to write down the general solution to the system in b.

```
> with(LinearAlgebra) :
> A := Matrix(3, 3, [-1/2, 0, 0, 1/2, -1/4, 0, 0, 1/4, -1/5]);
  Eigenvectors(A);
```

$$A := \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{5} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{3}{5} \\ -\frac{1}{5} & 0 & -\frac{6}{5} \\ 1 & 1 & 1 \end{bmatrix} \tag{2}$$

```
>
```

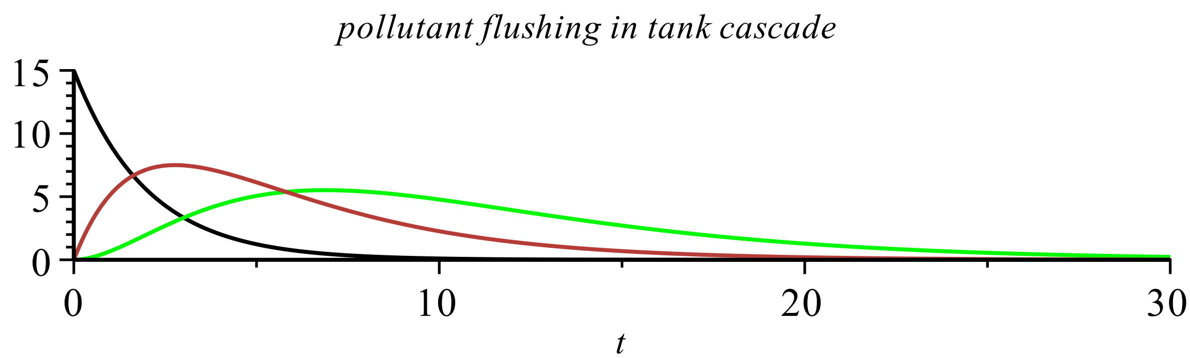
1d) Solve the IVP for this tank cascade, assuming that there are initially 15 *lb* of salt in the first tank, and no salt in the second and third tanks.

Your answer to d should be

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = 5\,\mathrm{e}^{-0.5\,t}\begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} + 30\,\mathrm{e}^{-0.25\,t}\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125\,\mathrm{e}^{-0.2\,t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

1e) We can plot the amounts of salt in each tank to figure out what's going on. Make sure you understand how the formulas below are related to the vector equation above, and interpret the graphical results.

```
> with(plots):
> x1 := t→15·exp(−.5·t):
  plot1 := plot(x1(t), t=0..30, color=black):
  x2 := t→ −30·exp(−.5·t) + 30·exp(−.25·t):
  plot2 := plot(x2(t), t=0..30, color=brown):
  x3 := t→25·exp(−.5·t) − 150·exp(−.25·t) + 125·exp(−.2·t):
  plot3 := plot(x3(t), t=0..30, color=green):
  display({plot1, plot2, plot3}, title=`pollutant flushing in tank cascade`);
```




```
>
```

Exercise 2) Use the same tank cascade. Only now, assume that there is initially 13 *lb* salt in the first tank, none in the others, and that when the water starts flowing the input pipe contains salty water, with concentration $0.5\ \frac{lb}{gal}$.

2a) Explain why this yields an IVP for an inhomogeneous system of linear DE's, namely

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ 0 \end{bmatrix}.$$

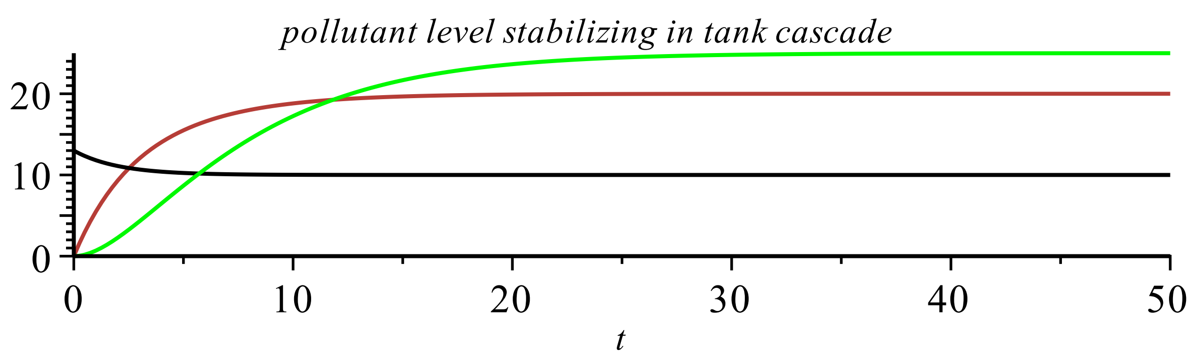
2b) Use a vector analog of "undetermined coefficients" to guess that there might be a particular solution that is a constant vector, i.e.

$$\underline{\boldsymbol{x}}_P(t) = \underline{\boldsymbol{c}}.$$

Plug this guess into the inhomgeneous system to deduce $\underline{\boldsymbol{c}}$. How could your irritating younger sibling have told you this particular solution, without knowing anything at all about linear algebra or differential equations?

2c) Use $\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_P(t) + \underline{\boldsymbol{x}}_H(t)$ to solve the IVP. Compare your solution to the plots below.

```
> x1 := t→10 + 3·exp(−.5·t) :
  plot1 := plot(x1(t), t=0 ..50, color=black) :
  x2 := t→20 − 6·exp(−.5·t)−14·exp(−.25·t) :
  plot2 := plot(x2(t), t=0 ..50, color=brown) :
  x3 := t→25 + 5·exp(−.5·t) + 70·exp(−.25·t) − 100·exp(−.2·t) :
  plot3 := plot(x3(t), t=0 ..50, color=green) :
  display({plot1, plot2, plot3}, title=`pollutant level stabilizing in tank cascade`);
```




```
>
```

Math 2280-001
Fri Mar 10

Friday: Friday's notes
→ 1st: lab tank problem.

5.3 phase diagrams for two linear systems of first order differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\vec{x}' = A\vec{x}$

Our goal is to understand how the (tangent vector field) phase portraits and solution curve trajectories are shaped by the eigendata of the matrix A. This discussion will be helpful in Chapter 6, when we discuss autonomous non-linear first order systems of differential equations, equilibrium points, and linearization near equilibrium points.

We will consider the cases of real eigenvalues and complex eigenvalues separately.

Real eigenvalues If the matrix $A_{2\times 2}$ is diagonalizable, i.e. if there exists a basis $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ of $\mathbb{R}^2$ consisting of eigenvectors of A), then let λ_1, λ_2 be the corresponding eigenvalues (which may or may not be distinct).

- In this case, the general solution to the system $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$ is
$$x(t) = c_1 e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\underline{\boldsymbol{v}}_2$$
- And, for each $\underline{\boldsymbol{x}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2$ the value of the tangent field at x is
$$A\,\underline{\boldsymbol{x}} = A\left(c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2\right) = c_1\lambda_1\underline{\boldsymbol{v}}_1 + c_2\lambda_2\underline{\boldsymbol{v}}_2.$$

(The text discusses the case of non-diaganalizable A. This can only happen if $det(A - \lambda I) = \left(\lambda - \lambda_1\right)^2$, but the dimension of the $\lambda = \lambda_1$ eigenspace is only one.)

Exercise 1) This is an example of what happens when A has two real eigenvalues of opposite sign. Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Find the eigendata for A, and the general solution to the first order system of DE's.
b) (On the next page) use just the eigendata to sketch the tangent vector field (on the first plot). Begin by sketching the two eigenspaces.
c) (On the next page) use just the general solutions to the DE system to sketch representative solution curves (on the second plot).
Your answers to b,c should be consistent.

a) $\begin{vmatrix} -3-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = (-3-\lambda)(4-\lambda) + 6 = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2) = 0$

$\lambda = 3, -2.$

$E_{\lambda=3}$: $\left[\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right]$ $\quad R_1/2$ $\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ soln: $e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$E_{\lambda=-2}$: $\left[\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right]$

$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ soln $e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\boxed{\vec{x}(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) $A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 A\vec{v}_1 + c_2 A\vec{v}_2$

$= c_1 3\vec{v}_1 + c_2(-2\vec{v}_2)$

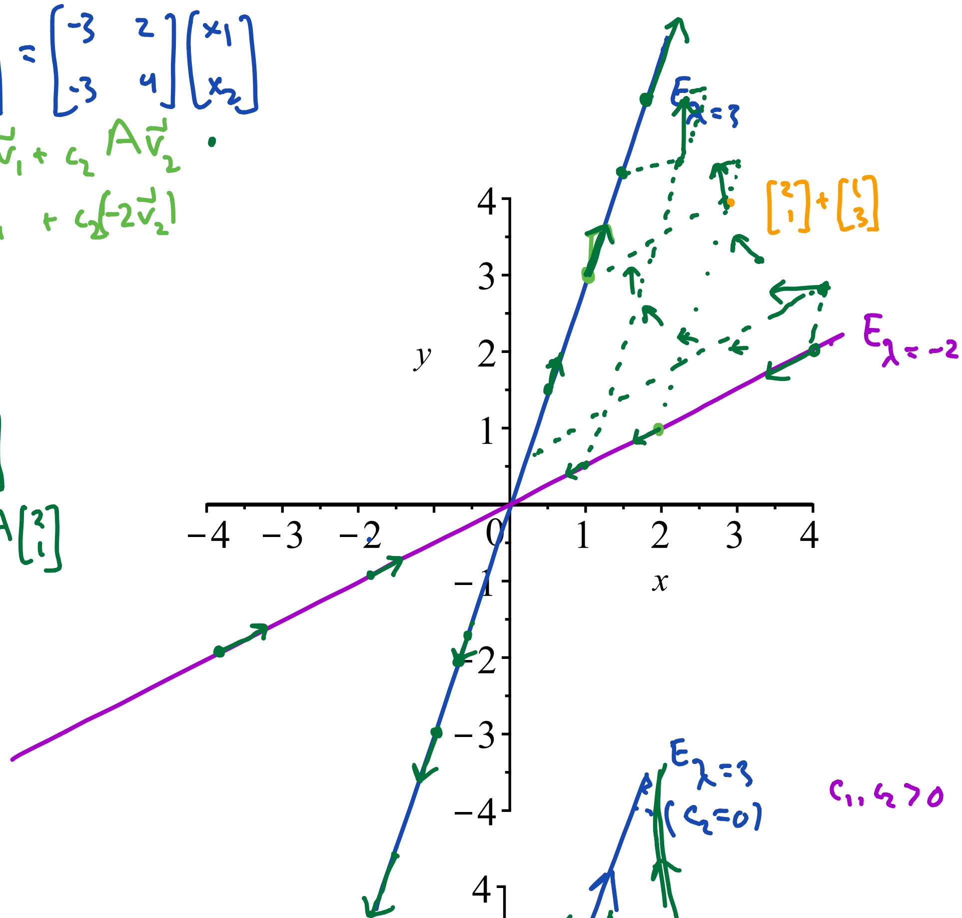
$A\begin{bmatrix}1\\3\end{bmatrix} = 3\begin{bmatrix}1\\3\end{bmatrix}$

$A\begin{bmatrix}2\\1\end{bmatrix} = -2\begin{bmatrix}2\\1\end{bmatrix}$

$A\left(c\begin{bmatrix}2\\1\end{bmatrix}\right) = c A\begin{bmatrix}2\\1\end{bmatrix}$

$A(c\vec{v}) = c A(\vec{v})$

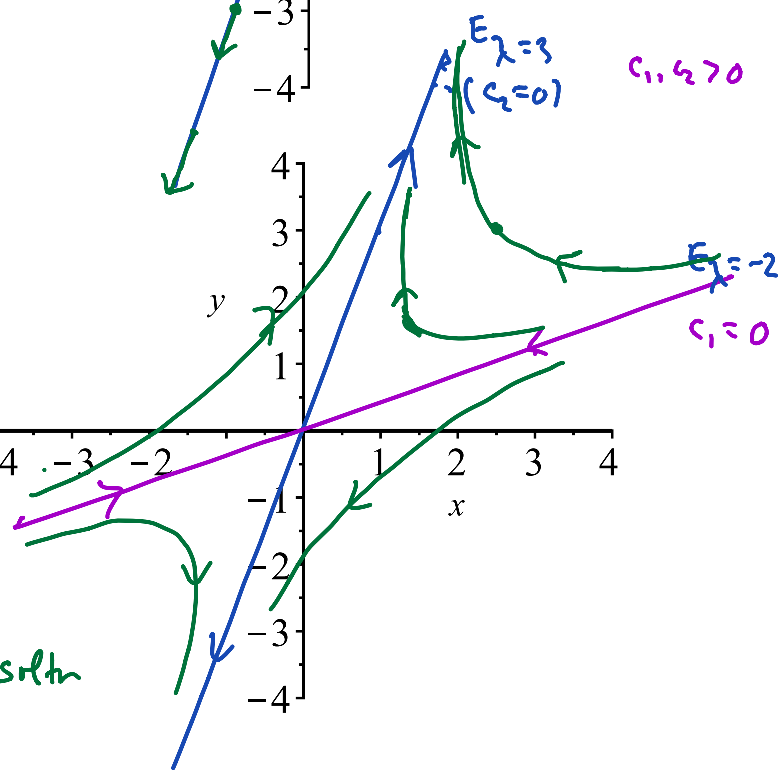
tedious!

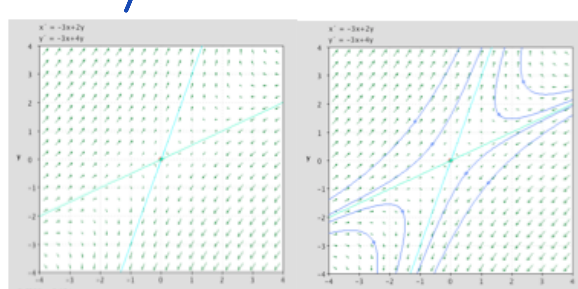


$\vec{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\3\end{bmatrix} + c_2 e^{-2t}\begin{bmatrix}2\\1\end{bmatrix}$

$\lambda_1 < 0 < \lambda_2$

origin is called saddle pt. equilibrium soln





Exercise 2) This is an example of what happens when A has two real eigenvalues of the same sign. Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Find the eigendata for A, and the general solution to the first order system of DE's.
b) Use the eigendata and the general solutions to construct a phase plane portrait of typical solution curves.

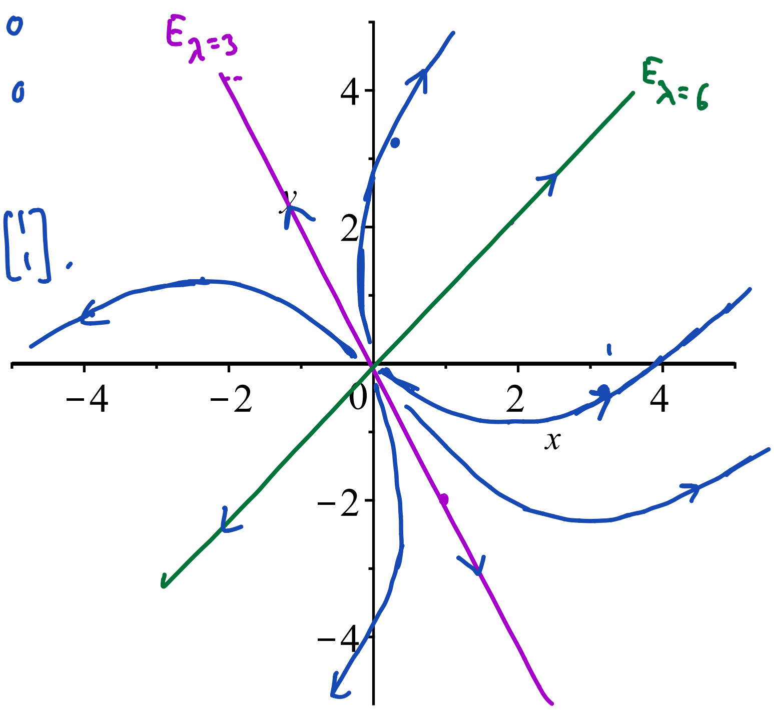
a) $\begin{vmatrix} 5-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (5-\lambda)(4-\lambda)-2 = \lambda^2-9\lambda+18 = (\lambda-6)(\lambda-3)$

$E_{\lambda=3}$ $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right]$ $\quad$ $E_{\lambda=6}$ $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 2 & -2 & 0 \end{array}\right]$

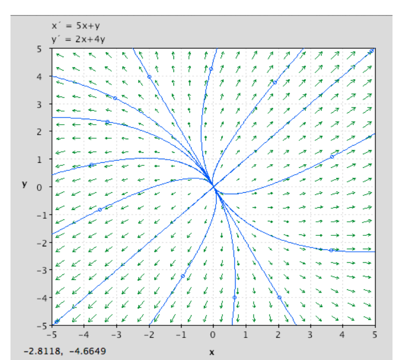
$\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad$ $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

soln $e^{3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad$ soln $e^{6t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{6t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$\lambda_1, \lambda_2 > 0$
(unstable) node.



Theorem: Time reversal: If $x(t)$ solves

$$x' = A\,x$$

then $z(t) := x(-t)$ solves

$$z' = (-A)z$$

proof: by the chain rule,

$$z'(t) = x'(-t)\cdot(-1) = -x'(-t) = -A\,x(-t) = -A\,z.$$

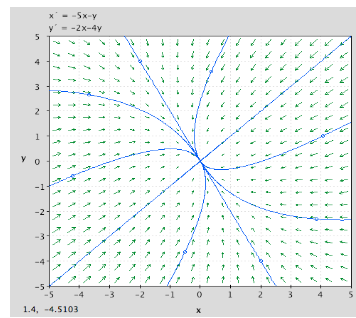
Exercise 3)

a) Let A be a square matrix, and let c be a scalar. How are the eigenvalues and eigenspaces of cA related to those of A?

b) Describe how the eigendata of the matrix in the system below, is related to that of the (opposite) matrix in the previous exercise. Also describe how the phase portraits are related.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\lambda_1, \lambda_2 < 0$



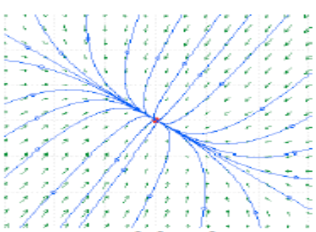
summary: In case the matrix $A_{2\times 2}$ is diagonalizable with real number eigenvalues, the first order system of DE's

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

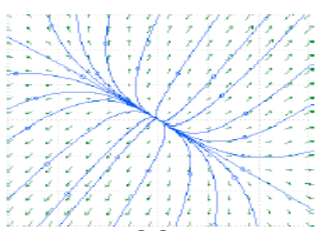
has general solution

$$\underline{\boldsymbol{x}}(t) = c_1 e^{\lambda_1 t}\underline{\boldsymbol{v}}_1 + c_2 e^{\lambda_2 t}\underline{\boldsymbol{v}}_2\,.$$

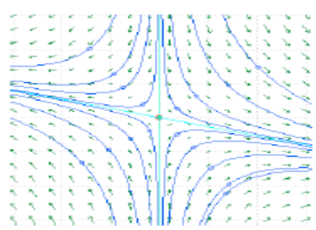
If each eigenvalue is non-zero, the three possibilities are:



nodal sink
$\lambda_1, \lambda_2 < 0$



nodal source
$\lambda_1, \lambda_2 > 0$



saddle point
$\lambda_1 < 0 < \lambda_2$

complex eigenvalues: Consider the first order system

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

Let $A_{2\times 2}$ have complex eigenvalues $\lambda = p \pm q\,i$. For $\lambda = p + q\,i$ let the eigenvector be $\underline{\boldsymbol{v}} = \underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i$.

Then we know that we can use the complex solution $e^{\lambda t}\underline{\boldsymbol{v}}$ to extract two real vector-valued solutions, by taking the real and imaginary parts of the complex solution

$$\begin{aligned}\underline{\boldsymbol{z}}(t) = e^{\lambda t}\underline{\boldsymbol{v}} &= e^{(p+q\,i)t}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i)\\ &= e^{p\,t}(\cos(q\,t) + i\sin(q\,t))(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}}\,i)\\ &= \left[e^{p\,t}\cos(q\,t)\underline{\boldsymbol{a}} - e^{p\,t}\sin(q\,t)\underline{\boldsymbol{b}}\right]\\ &\quad + i\left[e^{p\,t}\sin(q\,t)\underline{\boldsymbol{a}} + e^{p\,t}\cos(q\,t)\underline{\boldsymbol{b}}\right].\end{aligned}$$

$\vec{z}\,' = A\vec{z}$

$\vec{x}\,' + i\vec{y}\,' = A x + i A y$

Thus, the general real solution is a linear combination of the real and imaginary parts of the solution above:

$$\begin{aligned}\underline{\boldsymbol{x}}(t) &= c_1 e^{p\,t}[\cos(q\,t)\underline{\boldsymbol{a}} - \sin(q\,t)\underline{\boldsymbol{b}}]\\ &\quad + c_2 e^{p\,t}[\sin(q\,t)\underline{\boldsymbol{a}} + \cos(q\,t)\underline{\boldsymbol{b}}].\end{aligned}$$

$$= e^{pt}\left[\vec{a}\;\middle|\;\vec{b}\right]\begin{bmatrix} c_1\cos qt + c_2 \sin qt \\ -c_1 \sin qt + c_2 \cos qt\end{bmatrix}$$

We can rewrite $\underline{\boldsymbol{x}}(t)$ as

$$\underline{\boldsymbol{x}}(t) = e^{p\,t}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2\end{bmatrix}\begin{bmatrix}\cos(q\,t) & \sin(q\,t)\\ -\sin(q\,t) & \cos(q\,t)\end{bmatrix}\begin{bmatrix} c_1 \\ c_2\end{bmatrix}.$$

Breaking that expression down from right to left, what we have is:

$$\mathrm{Rot}_\theta = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

- parametric circle of radius $\sqrt{c_1^2 + c_2^2}$, with angular velocity $\omega = -q$:

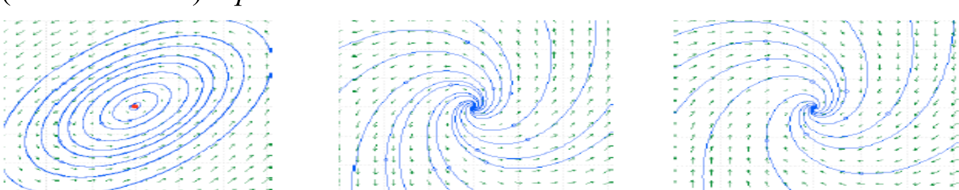
$$\begin{bmatrix}\cos(q\,t) & \sin(q\,t)\\ -\sin(q\,t) & \cos(q\,t)\end{bmatrix}\begin{bmatrix} c_1 \\ c_2\end{bmatrix}.$$

- transformed into a parametric ellipse by a matrix transformation of $\mathbb{R}^2$:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2\end{bmatrix}\begin{bmatrix}\cos(q\,t) & \sin(q\,t)\\ -\sin(q\,t) & \cos(q\,t)\end{bmatrix}\begin{bmatrix} c_1 \\ c_2\end{bmatrix}.$$

- possibly transformed into a shrinking or growing spiral by the scaling factor $e^{p\,t}$, depending on whether $p < 0$ or $p > 0$. If $p = 0$, curve remains an ellipse.

Thus $\underline{\boldsymbol{x}}(t)$ traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$:



center $\mathrm{Re}(\lambda)=0$ | spiral source $\mathrm{Re}(\lambda)>0$ | spiral sink $\mathrm{Re}(\lambda)<0$

Exercise 4) Do the eigendata analysis, find the general solution, and use tangent vectors just along the two axes to sketch typical solution curve trajectories, for this system from your homework due today:

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = \lambda(\lambda+2)+5$$
$$= \lambda^2+2\lambda+5$$
$$= (\lambda+1)^2+4 = 0$$
$$(\lambda+1)^2 = -4$$
$$\lambda+1 = \pm 2i$$
$$\lambda = -1 \pm 2i$$