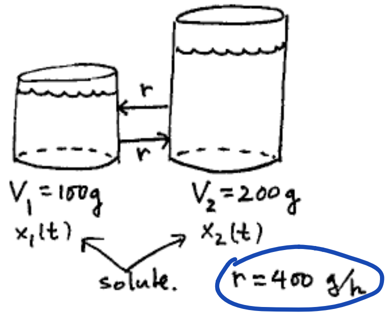
Quiz 9-9:15
9:15-9:25 for last HW problem

Math 2280-001
Fri Mar 3

4.1 Systems of differential equations - to model multi-component systems via compartmental analysis

http://en.wikipedia.org/wiki/Multi-compartment_model
and to understand higher order differential equations.

Here's a relatively simple 2-tank problem to illustrate the ideas:




Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$$x_1'(t) = r_i c_i - r_o c_o$$
$$= 400 \cdot \frac{x_2}{200} - 400 \frac{x_1}{100} = 2x_2 - 4x_1 = -4x_1 + 2x_2$$
$$\left(\frac{g}{h}\right)\left(\frac{\text{mass}}{g}\right)$$

$$x_2'(t) = r_i c_i - r_o c_o$$
$$= 400 \frac{x_1}{100} - 400 \frac{x_2}{200} = 4x_1 - 2x_2$$

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

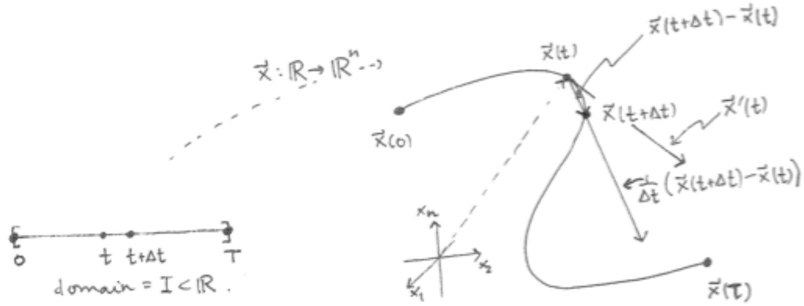
The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

$$\boxed{\begin{aligned} \underline{\boldsymbol{x}}'(t) &= \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t)) \\ \underline{\boldsymbol{x}}(t_0) &= \underline{\boldsymbol{x}}_0 \end{aligned}}$$

- We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

Why we expect IVP's for first order systems of DE's to have unique solutions $\underline{\boldsymbol{x}}(t)$ :

- From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{\boldsymbol{x}}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{\boldsymbol{x}}(t)$, as $t$ varies. This picture should remind you of the discussion, but ask questions if this is new to you:




Analytically, the reason that the vector of derivatives $\underline{\boldsymbol{x}}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\underline{\boldsymbol{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ \vdots \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}\right)$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$), and you know your "velocity" vector (depending on time and current location) $\Rightarrow$ you expect a unique solution! (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! And this is what numerical solvers do.)

These are vector analogs of the theorems we discussed in Chapter 1 for first order scalar differential equations. The first one should make intuitive sense, based on the reasoning of the previous page.

Theorem 1 For the IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}})$ is continuous in the $t$-variable and differentiable in its $\underline{\boldsymbol{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

$\vec{x}'(t) - A(t)\vec{x}(t) = \vec{f}(t)$

[chptr 1 $x' + P(t)x = Q(t)$]

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval $I$ containing $t_0$ then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. For example, with time-step $h$ the Euler loop would increment as follows:

$$t_{j+1} = t_j + h$$
$$\underline{\boldsymbol{x}}_{j+1} = \underline{\boldsymbol{x}}_j + h\,\underline{\boldsymbol{F}}(t_j, \underline{\boldsymbol{x}}_j).$$

Remark: These theorems are the true explanation for why the $n^{th}$-order linear DE IVPs in Chapter 3 always have solutions - We will see that each $n^{th} - order$ linear DE IVP is actually equivalent to an IVP for a first order system of $n$ linear DE's. (The converse is not true.) In fact, when software finds numerical approximations for solutions to higher order (linear or non-linear) DE IVPs that can't be found by the techniques of Chapter 3 or other mathematical formulas, it converts these IVPs to the equivalent first order system IVPs, and uses algorithms like Euler and Runge-Kutta to approximate the solutions.

Exercise 2) Return to the page 1 tank example

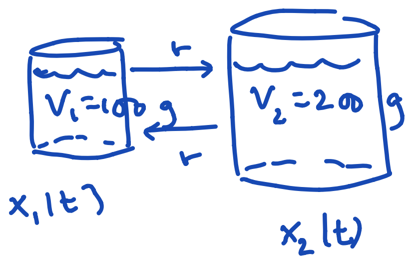
$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4x_1 + 2x_2 \\ 4x_1 - 2x_2 \end{bmatrix} = \begin{matrix} (4x_1 - 2x_2) \\ 2(2x_1 - x_2) \end{matrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x_1(0) = 9$$
$$x_2(0) = 0$$

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

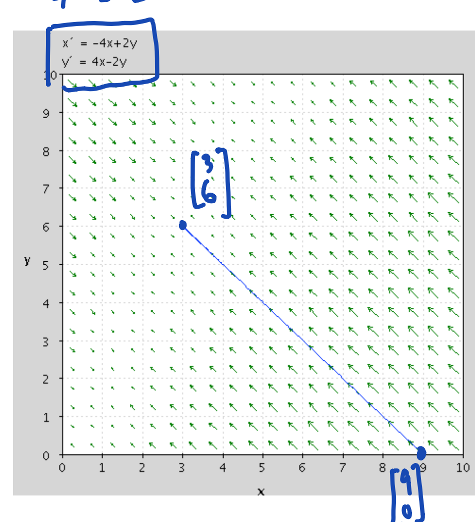
2b) What are the apparent limiting solute amounts in each tank? $\rightarrow$ $x_1(t) \to 3$, $x_2(t) \to 6$

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?




$r = 400 \frac{g}{h}$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$




$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

First order systems of differential equations of the form

$$\underline{x}'(t) = A(t)\,\underline{x}$$

are called linear homogeneous systems of DE's. (Think of rewriting the system as

$$\boxed{\underline{x}'(t) - A(t)\,\underline{x} = \underline{0}}$$

in analogy with how we wrote linear scalar differential equations. ) Then the inhomogeneous system of first order DE's would be written as

$$\boxed{\underline{x}'(t) - A(t)\,\underline{x} = \underline{f}(t)}$$

or

$$\underline{x}'(t) = A(t)\,\underline{x} + \underline{f}(t)$$

Exercise 3a) Show the space of solutions $\underline{x}(t)$ to the homogeneous system of DE's

$$\underline{x}'(t) - A(t)\,\underline{x} = \underline{0}$$

is a subspace, i.e. linear combinations of solutions are solutions.

$L(\vec{x}(t)) = x'(t) - A(t)\vec{x}(t)$

$V = \{\vec{x}(t): I \to \mathbb{R}^n,\ \text{s.t. } \vec{x}(t) \text{ is continuous},\ \vec{x}'(t) \text{ is continuous}\}$

Check $L$ is linear

$L: V \to W$

$W = \{\vec{y}(t): I \to \mathbb{R}^n \text{ s.t. } \vec{y}(t) \text{ is cont}\}$

(i) $L(\vec{x}(t) + \vec{z}(t)) = L(\vec{x}(t)) + L(\vec{z}(t))$

$\parallel$

$(x' + z') - A(x+z)$

$= x' + z' - Ax - Az$

$= (x' - Ax) + (z' - Az)$

$= L(x) + L(z)$

(ii) $L(c\vec{x}(t))$

$= (c\vec{x})' - Ac\vec{x}$

$= c(\vec{x}' - A\vec{x})$

$= c\,L(\vec{x})$

3b) In the special case that $A$ is a constant matrix ("constant coefficients"), look for a basis of solutions of the form

$$\underline{x}(t) = e^{\lambda t}\underline{v},$$

where $\underline{v}$ is a constant vector. Hint: In order for such an $\underline{x}(t)$ to solve the DE it must be true that

$$\underline{x}'(t) = \lambda e^{\lambda t}\underline{v}$$

and

$$A\,\underline{x}(t) = A\,e^{\lambda t}\underline{v} = e^{\lambda t}A\,\underline{v}$$

must agree. *These functions of t will agree if and only* $\lambda\,\underline{v} = A\,\underline{v}$. So, it's time to recall eigenvalues and eigenvectors! (Math 2270).

$\lambda\vec{v} = A\vec{v}$

get soltns!

$A\vec{v} - \lambda\vec{v} = \vec{0}$

$A\vec{v} - \lambda I\vec{v} = \vec{0}$

$\boxed{(A - \lambda I)\vec{v} = \vec{0}}$

3c) Solve the initial value problem of Exercise 2!! Compare your solution $\underline{x}(t)$ to the parametric curve on the previous page.

$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}. \end{cases}$$

1

eigenvalues & eigenvectors

$|A - \lambda I| = 0$

$\begin{vmatrix} -4-\lambda & 2 \\ 4 & -2-\lambda \end{vmatrix} = (\lambda+4)(\lambda+2) - 8$

$= \lambda^2 + 6\lambda + \cancel{8} - \cancel{8}$

$= \lambda(\lambda+6) = 0$

$\lambda = 0, -6.$

$E_{\lambda=0}$ $(A - 0I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$R_1/2$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

soln $e^{\lambda t}\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$(e^{0t}\vec{v})$

$E_{\lambda=-6}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

soln $e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

IVP: $\begin{bmatrix} 9 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -9 \\ -18 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

@ $t=0$: $\begin{bmatrix} 9 \\ 0 \end{bmatrix}$ ✓

long way.

$$\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}$$

$v_1 = \frac{1}{2}t$

$v_2 = t$

$\vec{v} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

Exercise 4) Lessons learned from Exercise 3:

a) What condition on the constant matrix $A_{n \times n}$ will allow you to solve every initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}$$
$$\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$$

using the method in Exercise 3 ? Hint: Math 2270 discussions (If that condition fails there are other ways to find the unique solutions.)

short way.

$A\vec{v} = v_1 \operatorname{col}_1(A) + v_2 \operatorname{col}_2(A) + \dots + v_n \operatorname{col}_n(A)$

$1 \cdot \operatorname{col}_1(A) + 2\operatorname{col}_2(A) = \vec{0}$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{u}(t) = e^{0t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}(t) = e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

are a basis for the sol'n space to $\vec{x}' = A\vec{x}$ is #3.

(1) span: Can solve any IVP with $\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$

at $t=0$: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

and each IVP only has one sol'n (any sol'n has initial vector)

so every sol't. is a linear combo.

b) What is the dimension of the solution space to the first order system

$$\mathbf{x}'(t) = A\mathbf{x}$$

when $\mathbf{x}_0 \in \mathbb{R}^n$ and $A = A_{n \times n}$ ?

(2) independence.

$c_1\vec{u}(t) + c_2\vec{v}(t) \equiv 0$

@ $t=0$. $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = 0.$

Exercise 7) Consider an $n^{th}$ order differential equation for a function $x(t)$:

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \ldots x^{(n-1)}(t)\right)$$

and the IVP

$$\begin{gathered} x^{(n)} = f\left(t, x, x', \ldots x^{(n-1)}\right) \\ x(t_0) = b_0 \\ x'(t_0) = b_1 \\ x''(t_0) = b_2 \\ \vdots \\ x^{(n-1)}(t_0) = b_{n-1}. \end{gathered}$$

a) Show that if $x(t)$ solves the IVP above, and if we define functions $x_1(t), x_2(t), \ldots x_n(t)$ by

$$x_1 := x,\ x_2 := x',\ x_3 := x'',\ \ldots\ x_n := x^{(n-1)}$$

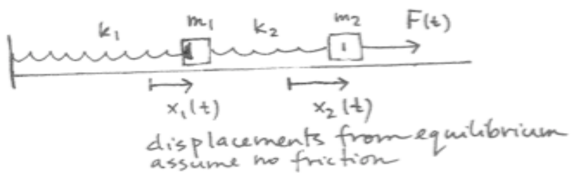
Then $[x_1, x_2, \ldots, x_n]$ solve the first order system IVP

$$\begin{gathered} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = f\left(t, x_1, x_2, \ldots x_{n-1}\right) \\ x_1(t_0) = b_0 \\ x_2(t_0) = b_1 \\ x_3(t_0) = b_2 \\ \vdots \\ x_n(t_0) = b_{n-1} \end{gathered}$$

b) Show that if $[x_1(t), x_2(t), \ldots x_n(t)]$ is a solution to the IVP in a, then the first function $x_1(t)$ solves the original IVP for the $n^{th}$ order differential equtation.

Higher order systems of DE's are also equivalent to first order systems, as illustrated in the next example.

Consider this configuration of two coupled masses and springs:




Exercise 8) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case?

Exercise 9) Consider the IVP from Exercise 6, with the special values
$m_1 = 2, m_2 = 1; k_1 = 4, k_2 = 2; F(t) = 40 \sin(3\,t)$ :

$$
\begin{aligned}
&x_1'' = -3\,x_1 + x_2\\
&x_2'' = 2\,x_1 - 2\,x_2 + 40 \sin(3\,t)\\
&x_1(0) = b_1, x_1'(0) = b_2\\
&x_2(0) = c_1, x_2'(0) = c_2 .
\end{aligned}
$$

9a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define

$$
\begin{aligned}
v_1(t) &:= x_1'(t)\\
v_2(t) &:= x_2'(t)
\end{aligned}
$$

then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

$$
\begin{aligned}
&x_1' = v_1\\
&x_2' = v_2\\
&v_1' = -3\,x_1 + x_2\\
&v_2' = 2\,x_1 - 2\,x_2 + 40 \sin(3\,t)\\
&x_1(0) = b_1\\
&v_1(0) = b_2\\
&x_2(0) = c_1\\
&v_2(0) = c_2 \quad .
\end{aligned}
$$

9b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.