• posted review notes on CANVAS

Math 2280-001 Week 11
Mar 27-29 (Exam 2 on Mar 31)

Mon Mar 27: Use last Friday's notes to discuss matrix exponentials

Wed Mar 29: 5.6-5.7 Matrix exponentials, linear systems, and variation of parameters for inhomogeneous systems.

Recall: For the first order system

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

• $\Phi(t)$ is a fundamental matrix (FM) if its n columns are a basis for the solution space to the first order system above (i.e. $\Phi(t)$ is the Wronskian matrix for a basis to the solution space).

• $\Phi(t)$ is an FM if and only if $\underline{\Phi'(t) = A\,\Phi}$ and $\underline{\Phi(0)}$ is invertible.   C

• e^{tA} is the unique matrix solution to

$$\left.\begin{aligned} X'(t) &= A\,X \\ X(0) &= I \end{aligned}\right\}$$

and may be computed either of two ways:

$$e^{tA} = \Phi(t)\Phi(0)^{-1}$$

where $\Phi(t)$ is any other FM, or via the infinite series

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots + \frac{t^k}{k!}A^k + \ldots$$

Example 1: We showed that if

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \ldots & 0 \\ 0 & \lambda_2 & 0 \ldots & 0 \\ : & : & & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

then

$$e^{t\Lambda} = \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2}{2!}\lambda_1^2 + \ldots & 0 & 0 \ldots & 0 \\ 0 & 1 + t\lambda_2 + \frac{t^2}{2!}\lambda_2^2 + \ldots & 0 \ldots & 0 \\ : & : & & : \\ 0 & 0 & \ldots & 1 + t\lambda_n + \frac{t^2}{2!}\lambda_n^2 + \ldots \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} e^{t\lambda_1} & 0 & \dots & 0 \\ 0 & e^{t\lambda_2} & \dots & 0 \\ : & : & : & : \\ 0 & 0 & 0 & e^{t\lambda_n} \end{bmatrix}$$

$$A[\vec{v}_1|\vec{v}_2|\dots] = [\vec{v}_1|\vec{v}_2 \dots]\begin{bmatrix}\lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\end{bmatrix}$$

$AS = S\Lambda$ ✓

$\underline{S^{-1}AS = \Lambda}$, $A = S\Lambda S^{-1}$

<u>Example 2)</u> If A is diagonalizable we showed that we got the same answer for e^{tA} using

$$\underline{A = S\Lambda S^{-1}}$$

and the Taylor series method, as we did we did using the $\Phi(t)\Phi(0)^{-1}$ method. In fact, $\Phi(t) = S\,e^{t\Lambda}$ and $\Phi(0)^{-1} = S^{-1}$, so this makes sense.

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \dots$$

$$= SS^{-1} + tS\Lambda S^{-1} + \frac{t^2}{2}\underbrace{(S\Lambda S^{-1})^2}_{\substack{S\Lambda \underbrace{S^{-1}S}_{I}\Lambda S^{-1} \\ S\Lambda^2 S^{-1}}} + \frac{t^3}{3!}\underbrace{(S\Lambda S^{-1})^3}_{S\Lambda^3 S^{-1}} + \dots$$

$$= S\left[I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \frac{t^3}{3!}\Lambda^3 + \dots\right]S^{-1}$$

$$e^{tA} = S\,e^{t\Lambda}\,S^{-1}$$

$$e^{tA} = S\begin{bmatrix} e^{t\lambda_1} & & 0 \\ & e^{t\lambda_2} & \\ 0 & & \ddots \; e^{t\lambda_n}\end{bmatrix}S^{-1} \leftarrow e^{tA} \text{ using power series}$$

same answer

$$e^{tA} = \underbrace{\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \dots \vec{v}_n\right]\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t}\end{bmatrix}}\; S^{-1}$$

$$e^{tA} = \left[e^{\lambda_1 t}\vec{v}_1 \;\middle|\; e^{\lambda_2 t}\vec{v}_2 \;\middle|\; \dots \;\middle|\; e^{\lambda_n t}\vec{v}_n \right]\Phi(0)^{-1} \leftarrow e^{tA} \text{ using } \Phi(t)\Phi(0)^{-1}$$

$\Phi(t)$ use eigenvalues & eigenvectors.

How to compute e^{At} when A is not diagonalizable. This method depends on the fundamental fact about how the generalized eigenspaces of a matrix fit together.

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"Recall" (this is really linear algebra material, but most of you haven't seen it.)

For $A_{n \times n}$ let the characteristic polynomial $p(\lambda) = det(A - \lambda I)$ factor as

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \ldots (\lambda - \lambda_m)^{k_m}$$

Any eigenspace of A for which $dim\left(E_{\lambda_j}\right) < k_j$ is called defective. If A has any defective eigenspaces then it is not diagonalizable. (If none of the eigenspaces are defective, then by amalgamating bases for each eigenspace one obtains a basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$, in the case of complex eigendata). However, the larger generalized eigenspaces G_{λ_j} defined by

$$\text{nullspace}(A-\lambda_j I) = E_{\lambda_j} \subseteq G_{\lambda_j} := nullspace\left(\left(A - \lambda_j I\right)^{k_j}\right)$$

do always have dimension k_j. If bases for each G_{λ_j} are amalgamated they will form a basis for $\mathbb{R}^n$ or $\mathbb{C}^n$.

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For each basis vector $\underline{\boldsymbol{v}}$ of G_{λ_j} one can construct a basis solution $x(t)$ to $x' = A\,x$ as follows:

$e^{B+C} = e^B e^C$ when B & C commute

$A = \lambda_j I + (A-\lambda_j I)$

$At = \underbrace{\lambda_j I t + (A-\lambda_j I)t}_{\text{commute.}}$

$$\underline{\boldsymbol{x}}(t) = e^{At}\underline{\boldsymbol{v}} = e^{\lambda_j I t} e^{(A - \lambda_j I)t}\underline{\boldsymbol{v}}$$

$$= e^{\lambda_j t} I \left(I + t\left(A - \lambda_j I\right) + \frac{t^2}{2!}\left(A - \lambda_j I\right)^2 + \ldots\right)\underline{\boldsymbol{v}}$$

$$= e^{\lambda_j t}\left(\underline{\boldsymbol{v}} + t\left(A - \lambda_j I\right)\underline{\boldsymbol{v}} + \frac{t^2}{2!}\left(A - \lambda_j I\right)^2\underline{\boldsymbol{v}} + \ldots\right)$$

$$= e^{\lambda_j t}\left(\underline{\boldsymbol{v}} + t\left(A - \lambda_j I\right)\underline{\boldsymbol{v}} + \frac{t^2}{2!}\left(A - \lambda_j I\right)^2\underline{\boldsymbol{v}} + \frac{t^{k_j - 1}}{(k_j - 1)!}\left(A - \lambda_j I\right)^{k_j - 1}\underline{\boldsymbol{v}} + 0 + 0 + \ldots\right)$$

$$\underline{\boldsymbol{x}}(t) = e^{\lambda_j t}\left[\underline{\boldsymbol{v}} + t\left(A - \lambda_j I\right)\underline{\boldsymbol{v}} + \frac{t^2}{2!}\left(A - \lambda_j I\right)^2\underline{\boldsymbol{v}} + \frac{t^{k_j - 1}}{(k_j - 1)!}\left(A - \lambda_j I\right)^{k_j - 1}\underline{\boldsymbol{v}}\right].$$

Notes: The final sum is a finite sum because $\underline{\boldsymbol{v}} \in nullspace\left(\left(A - \lambda_j I\right)^{k_j}\right)$! If $\underline{\boldsymbol{v}}$ was an eigenvector, you've just reconstructed

$$\underline{\boldsymbol{x}}(t) = e^{\lambda_j t}\underline{\boldsymbol{v}}$$

Use the n independent solutions found this way to construct a $\Phi(t)$, and compute $e^{At} = \Phi(t)\Phi(0)^{-1}$.

Exercise 1 Find e^{At} for the matrix in the system:

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\vec{x}' = A\vec{x}.$

$$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

$E_{\lambda=2}$: $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_{\lambda=2} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$\lambda=2$ is defective.

(dim = 1, alg mult = 2)

soln: $\vec{x}(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\Phi(t) = \begin{bmatrix} \vec{x}(t) & \vec{y}(t) \end{bmatrix}$$

$$\Phi(t) = e^{2t}\begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix}$$

$$e^{At} = \Phi(t)\,\Phi(0)^{-1}$$

$$= e^{2t}\begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

$$= e^{2t}\begin{bmatrix} 1 & 1+t \\ 1 & t \end{bmatrix}\frac{1}{+1}\begin{bmatrix} 0 & +1 \\ +1 & -1 \end{bmatrix}$$

$$e^{At} = e^{2t}\begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}$$

Theorem about G_λ says $\dim G_{\lambda=2} = 2$

$$G_{\lambda=2} = \text{nullspace}\left((A-2I)^2\right).$$

$$(A-2I)^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so $G_{\lambda=2} = \mathbb{R}^2$

for 2nd soln $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (any $\vec{v}$ independent of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ would work)

use previous page.

$\vec{y}(t) = e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a soln.

$$= e^{(2I + (A-2I))t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$= e^{2It}\,\underline{e^{(A-2I)t}}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= e^{2t}I\left(I + (A-2I)t + (A-2I)^2\frac{t^2}{2} + \dots\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e^{t\Lambda} = \begin{bmatrix} e^{t\lambda_1} & & 0 \\ & e^{t\lambda_2} & \\ 0 & & \ddots \end{bmatrix}$$

$$= e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t(A-2I)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cancel{\frac{t^2}{2!}(A-2I)^2\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \cancel{+ \dots}\right)$$

$$= e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$= e^{2t}\begin{bmatrix} 1+t \\ t \end{bmatrix}$$

$\vec{y}(t)$

tech check:

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> with(LinearAlgebra) :
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$$A := \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix};$$

$factor(Determinant(A - \lambda \cdot IdentityMatrix(2)));$

$$A := \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

$$(\lambda - 2)^2 \qquad \textbf{(1)}$$

So the only eigenvalue is $\lambda = 2$.

$> B := A - 2 \cdot IdentityMatrix(2);$

$NullSpace(B);$

$$B := \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \qquad \textbf{(2)}$$

but $E_{\lambda=2}$ is only one-dimensional. However, the generalized eigenspace $G_{\lambda=2} = nullspace(A - 2\,I)^2$ will be two dimensional:

$> NullSpace(B^2);$

$$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \qquad \textbf{(3)}$$

Use this generalized nullspace basis to construct a basis of solutions to $x' = A\,x$ and use the resulting $\Phi(t)$ to construct the matrix exponential...

$> x1 := t \rightarrow e^{2 \cdot t} \cdot (IdentityMatrix(2) + t \cdot B) . \begin{bmatrix} 1 \\ 0 \end{bmatrix} :$

$x1(t);$

$x2 := t \rightarrow e^{2 \cdot t} \cdot \left(IdentityMatrix(2) + t \cdot B + \frac{t^2}{2} \cdot B^2 \right) . \begin{bmatrix} 0 \\ 1 \end{bmatrix} :$

$x2(t);$

$\Phi := t \rightarrow \langle x1(t) | x2(t) \rangle :$

$\Phi(t);$

$\Phi(t) . \Phi(0)^{-1};$

$MatrixExponential(t \cdot A);$ *#these last two should be the same!! In fact, the last three in this case*

$$\begin{bmatrix} e^{2t}\,(1 + t) \\ t\,e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} -t\,e^{2t} \\ (1-t)\,e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} e^{2t}\,(1+t) & -t\,e^{2t} \\ t\,e^{2t} & (1-t)\,e^{2t} \end{bmatrix} \qquad \underline{\Phi}(t)$$

$$\begin{bmatrix} e^{2t}\,(1+t) & -t\,e^{2t} \\ t\,e^{2t} & (1-t)\,e^{2t} \end{bmatrix} \qquad \underline{\Phi}(t)\,\underline{\Phi}(0)^{-1}$$

$$\begin{bmatrix} e^{2t}\,(1+t) & -t\,e^{2t} \\ t\,e^{2t} & -e^{2t}\,(-1+t) \end{bmatrix} \qquad e^{tA} \tag{4}$$

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>

<u>Variation of parameters:</u> This is what fundamental matrices and matrix exponentials are especially good for....they let you solve non-homogeneous systems without guessing. Consider the non-homogeneous first order system

$$\underline{\boldsymbol{x}}'(t) = P(t)\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t) \quad *$$

Let $\Phi(t)$ be an FM for the homogeneous system

$$\underline{\boldsymbol{x}}'(t) = P(t)\underline{\boldsymbol{x}}.$$

Since $\Phi(t)$ is invertible for all t we may do a change of functions for the non-homogeneous system:

$$\underline{\boldsymbol{x}}(t) = \Phi(t)\underline{\boldsymbol{u}}(t)$$

plug into the non-homogeneous system (*):

$$\Phi'(t)\underline{\boldsymbol{u}}(t) + \Phi(t)\underline{\boldsymbol{u}}'(t) = P(t)\Phi(t)\underline{\boldsymbol{u}}(t) + \underline{\boldsymbol{f}}(t).$$

Since $\Phi' = P\,\Phi$ the first terms on each side cancel each other and we are left with

$$\Phi(t)\underline{\boldsymbol{u}}'(t) = \underline{\boldsymbol{f}}(t)$$

$$\underline{\boldsymbol{u}}' = \Phi^{-1}\underline{\boldsymbol{f}}$$

which we can integrate to find a $\underline{\boldsymbol{u}}(t)$, hence an $\underline{\boldsymbol{x}}(t) = \Phi(t)\underline{\boldsymbol{u}}(t)$.

<u>Remark</u>: This is where the (mysterious at the time) formula for variation of parameters in n^{th} order linear DE's came from....

"Recall" (February 24 notes):

....

<u>Variation of Parameters:</u> The advantage of this method is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L, and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x), \ldots y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0\,.$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x), \ldots u_n(x)$ have derivatives satisfying the Wronskian matrix equation

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ : \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ : \\ f \end{bmatrix}$$

....

But if we convert the n^{th} order DE into a first order system for $x_1 = y, x_2 = y'$ etc. we have

$$\begin{aligned} x_1 &\quad (= y) \\ x_1' = x_2 &\quad (= y') \\ x_2' = x_3 &\quad (= y'') \\ x_{n-1}' = x_n &\quad \left(= y^{(n-1)}\right) \\ x_n' = \left(= y^{(n)}\right) &= -p_0(x)y_1 - p_1(x)y_2 - \ldots - p_{n-1}(x)y_{n-1} + f. \end{aligned}$$

And each basis solution $y(t)$ for $L(y) = 0$ gives a solution $\left[y, y', y'', \ldots y^{(n-1)}\right]^T$ to the homogeneous system

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & 1 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -p_0 & -p_1 & -p_2 & \ldots & -p_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f \end{bmatrix}.$$

So the original Wronskian matrix for the n^{th} order linear homogeneous DE is a FM for the system above, so the formula we learned in Chapter 3 is a special case of the easier to understand one for first order systems that we just derived, namely

$$\Phi(t)\underline{\boldsymbol{u}}'(t) = \underline{\boldsymbol{f}}(t)$$

$$\underline{\boldsymbol{u}}' = \Phi^{-1}\underline{\boldsymbol{f}}$$

Returning to first order systems, if we want to solve an IVP for a first order system rather than find the complete general solution, then the following two ways are appropriate:

<u>1) If you want to solve the IVP</u>

$$\begin{aligned} \underline{\mathbf{x}}'(t) &= P(t)\underline{\mathbf{x}} + \underline{\boldsymbol{f}}(t) \\ \underline{\mathbf{x}}(0) &= \underline{\mathbf{x}}_0 \end{aligned}$$

The the solution will be of the form $\underline{\mathbf{x}} = \Phi\,\underline{\boldsymbol{u}}$ (where $\underline{\boldsymbol{u}}' = \Phi^{-1}\underline{\boldsymbol{f}}$ as above). Thus

$$\underline{\mathbf{x}}_0 = \Phi(0)\underline{\boldsymbol{u}}_0$$

so

$$\underline{\boldsymbol{u}}_0 = \Phi(0)^{-1}\underline{\mathbf{x}}_0.$$

Thus

$$\underline{\boldsymbol{u}}(t) = \underline{\boldsymbol{u}}_0 + \int_0^t \underline{\boldsymbol{u}}'(s)\, ds$$

$$\underline{\boldsymbol{u}}(t) = \underline{\boldsymbol{u}}_0 + \int_0^t \underline{\boldsymbol{\Phi}}^{-1}(s)\underline{\boldsymbol{f}}(s)\, ds.$$

Then

$$\underline{\mathbf{x}}(t) = \Phi(t)\underline{\boldsymbol{u}}(t)$$

$$\underline{\mathbf{x}}(t) = \Phi(t)\left(\underline{\boldsymbol{u}}_0 + \int_0^t \underline{\boldsymbol{\Phi}}^{-1}(s)\underline{\boldsymbol{f}}(s)\, ds\right).$$

<u>2) If you want to solve the special case IVP</u>

$$\begin{cases} \underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}} + \underline{\boldsymbol{f}}(t) \\ \underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0 \end{cases}$$

where A is a constant matrix, you may derive a special case of the solution formula above just as we did in Chapter 1. This is sort of amazing!

$$\begin{aligned} \underline{\mathbf{x}}'(t) &= A\,\underline{\mathbf{x}} + \underline{\boldsymbol{f}}(t) \\ \underline{\mathbf{x}}'(t) - A\,\underline{\mathbf{x}} &= \underline{\boldsymbol{f}}(t) \\ e^{-tA}(\underline{\mathbf{x}}'(t) - A\,\underline{\mathbf{x}}) &= e^{-tA}\underline{\boldsymbol{f}}(t) \\ \int_0^t \Big(\ \frac{d}{dt}\left(e^{-tA}\underline{\mathbf{x}}(t)\right) &= e^{-tA}\underline{\boldsymbol{f}}(t)\,. \end{aligned}$$

$$\frac{d}{dt}\left(e^{-tA}\vec{x}(t)\right) = e^{-tA}(-A)\vec{x} + e^{-tA}\vec{x}' = e^{-tA}(-A\vec{x} + \vec{x}')$$

Integrate from 0 to t:

$$e^{-tA}\underline{\mathbf{x}}(t) - \underline{\mathbf{x}}_0 = \int_0^t e^{-sA}\underline{\boldsymbol{f}}(s)\, ds$$

Move the $\underline{\mathbf{x}}_0$ over and multiply both sides by e^{tA}:

$$\boxed{\underline{\mathbf{x}}(t) = e^{tA}\left(\underline{\mathbf{x}}_0 + \int_0^t e^{-sA}\underline{\boldsymbol{f}}(s)\, ds\right).}$$

Exercise 2 Consider the non-homogeneous problem related to the homogeneous system in Exercise 1:

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solve this system using the formula

$$\underline{\boldsymbol{x}}(t) = \mathrm{e}^{t A}\left(\underline{\boldsymbol{x}}_0 + \int_0^t \mathrm{e}^{-s A} \underline{\boldsymbol{f}}(s)\,\mathrm{d}s\right)$$

(One could also try undetermined coefficients, but variation of parameters requires no "guessing.")

Tech check: (The commands are sort of strange, but might help in your homework.)

> $with(LinearAlgebra):$

> $A := \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}:$

$MatrixExponential(t \cdot A);$

$f := t \rightarrow \begin{bmatrix} t \\ 0 \end{bmatrix}:$

$x0 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}:$

$$\begin{bmatrix} \mathrm{e}^{2t}(1+t) & -t\,\mathrm{e}^{2t} \\ t\,\mathrm{e}^{2t} & -\mathrm{e}^{2t}(-1+t) \end{bmatrix} \qquad \textbf{(5)}$$

> $integrand := s \rightarrow simplify(MatrixExponential(-s \cdot A).f(s)):$ $\#integrand\ in\ formula\ above$

> $integrand(t);$ $\#checking$

$$\begin{bmatrix} -e^{-2t}(-1+t)\,t \\ -t^2 e^{-2t} \end{bmatrix} \tag{6}$$

```
> integrated := unapply(map(int, integrand(s), s = 0 .. t), t) :  #"map"  applies a function to each entry of an array...
      # "unapply" makes a function out of output
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```
> integrated(t);  #checking
```

$$\begin{bmatrix} \frac{1}{2}t^2 e^{-2t} \\ -\frac{1}{4}+\frac{1}{4}e^{-2t}+\frac{1}{2}t\,e^{-2t}+\frac{1}{2}t^2 e^{-2t} \end{bmatrix} \tag{7}$$

```
> x := unapply(simplify(MatrixExponential(t·A).(x0 + integrated(t))), t) :
  x(t);  #checking answer
```

$$\begin{bmatrix} \frac{1}{4}t\left(e^{2t}-1\right) \\ \frac{1}{4}e^{2t}(-1+t)+\frac{1}{4}t+\frac{1}{4} \end{bmatrix} \tag{8}$$

```
> with(DEtools) :
  dsolve({x1'(t) = 3·x1(t) −x2(t) + t, x2'(t) = x1(t) + x2(t), x1(0) = 0, x2(0) = 0});
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$$\left\{x1(t)=\frac{1}{4}t\,e^{2t}-\frac{1}{4}t,\ x2(t)=-\frac{1}{4}e^{2t}+\frac{1}{4}+\frac{1}{4}t+\frac{1}{4}t\,e^{2t}\right\} \tag{9}$$

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>
>
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