Friday : finish mass-spring systems
quiz at end of class
Monday:
Tuesday: review session : go through old midterm.
Wed : continue w §5.7

Math 2280-001
Fri Mar 24
Do this on Monday.

5.6 Matrix exponentials and linear systems: The analogy between first order systems of linear differential equations (Chapter 5) and scalar linear differential equations (Chapter 1) is much stronger than you may have expected. This will become especially clear on Monday, when we study section 5.7.

Definition Consider the linear system of differential equations for $x(t)$:

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

$$\Phi(t) = \begin{bmatrix} \phi_{11}(t) & \cdots & \phi_{1n}(t) \\ \vdots & & \vdots \\ \phi_{n1}(t) & & \phi_{nn}(t) \end{bmatrix}$$

where $A_{n\times n}$ is a constant matrix as usual. If $\{\underline{\boldsymbol{x}}_1(t), \underline{\boldsymbol{x}}_2(t), \ldots \underline{\boldsymbol{x}}_n(t)\}$ is a basis for the solution space to this system, then the matrix having these solutions as columns,

$$\Phi(t) := \left[\underline{\boldsymbol{x}}_1(t) \middle| \underline{\boldsymbol{x}}_2(t) \middle| \ldots \middle| \underline{\boldsymbol{x}}_n(t)\right]$$

$$\Phi'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\Phi(t+\Delta t) - \Phi(t)\right]$$

is called a Fundamental Matrix (FM) to this system of differential equations. Notice that this equivalent to saying that $X(t) = \Phi(t)$ solves

$$\begin{cases} X'(t) = A\,X \\ X(0) \quad \text{nonsingular (i.e. invertible)} \end{cases}$$

$$= \lim_{\Delta t \to 0} \cancel{\frac{1}{\Delta t}} \begin{bmatrix} \frac{\phi_{11}(t+\Delta t) - \phi_{11}(t)}{\Delta t} & \cdots & \frac{\phi_{n1}(t+\Delta t) - \phi_{n1}(t)}{\Delta t} \\ \frac{\phi_{21}(t+\Delta t) - \phi_{21}(t)}{\Delta t} & & \\ \vdots & & \end{bmatrix}$$

(just look column by column). Notice that a FM is just the Wronskian matrix for a solution space basis.

$$= \begin{bmatrix} \phi_{11}'(t) & \cdots & \phi_{n1}'(t) \\ \phi_{21}'(t) & & \vdots \\ \vdots & & \\ \phi_{n1}'(t) & & \phi_{nn}'(t) \end{bmatrix}$$

so on

$$\Phi'(t) = \left[\vec{x}_1'(t) \middle| \vec{x}_2'(t) \middle| \cdots \middle| \vec{x}_n'(t)\right]$$

$$A\Phi = \left[A\vec{x}_1 \middle| A\vec{x}_2 \middle| \cdots \middle| A\vec{x}_n\right]$$

Example 1 page 351

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5)$$

$\lambda = -2$:

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \Rightarrow \underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\lambda = 5$:

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \Rightarrow \underline{\boldsymbol{v}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

general solution

$$\underline{\boldsymbol{x}}(t) = c_1 e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

$$e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Possible FM:

$$\Phi(t) = \left[\begin{array}{c|c} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{array}\right]$$

general solution:

$$\Phi(t)\underline{\boldsymbol{c}} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Theorem: If $\Phi(t)$ is a FM for the first order system $\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$ then the solution to

$$IVP \begin{cases} \underline{\boldsymbol{x}}'(\mathrm{t}) = A\,\underline{\boldsymbol{x}} \\ \underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \end{cases}$$

is

$$\boxed{\underline{\boldsymbol{x}}(t) = \Phi(t)\Phi(0)^{-1}\underline{\boldsymbol{x}}_0} = \Phi(t)\vec{c} \qquad \text{for } \vec{c} = \Phi(0)^{-1}\vec{x}_0$$

proof: Since $\underline{\boldsymbol{x}}(t) = \Phi(t)\Phi(0)^{-1}\underline{\boldsymbol{x}}_0 = \Phi(t)\left[\Phi(0)^{-1}\underline{\boldsymbol{x}}_0\right]$ is a linear combination of the columns of $\Phi(t)$ it is a solution to the homogeneous DE $\underline{\boldsymbol{x}}'(\mathrm{t}) = A\,\underline{\boldsymbol{x}}$. Its value at $t = 0$ is
$\underline{\boldsymbol{x}}(0) = \Phi(0)\Phi(0)^{-1}\underline{\boldsymbol{x}}_0 = \left[\Phi(0)\Phi(0)^{-1}\right]\underline{\boldsymbol{x}}_0 = I\,\underline{\boldsymbol{x}}_0 = \underline{\boldsymbol{x}}_0$.

□

Exercise 1) Continuing with the example on page 1, use the formula above to solve

$$IVP \begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

Adjoint formula for A^{-1}

$$A^{-1} = \frac{1}{|A|}\left(\mathrm{cof}(A)\right)^T$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \qquad \Phi(0) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \qquad \Phi(0)^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

IVP soln is

$$\Phi(t)\Phi(0)^{-1}\vec{x}_0 = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}\underbrace{\left(\frac{1}{7}\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\frac{1}{7}\begin{bmatrix} 3 \\ 2 \end{bmatrix}}\right)}_{\begin{bmatrix} 3/7 \\ 2/7 \end{bmatrix}}$$

$$\text{ans: } \underline{\boldsymbol{x}}(t) = \frac{3}{7}\mathrm{e}^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{2}{7}\mathrm{e}^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Remark: If $\Phi(t)$ is a Fundamental Matrix for $\underline{x}' = A\,\underline{x}$ and if C is an invertible matrix of the same size, then $\Phi(t)C$ is also a FM. Check: Does $X(t) = \Phi(t)C$ satisfy

more FM's !

$$\begin{cases} X'(t) = A\,X \\ X(0) \quad \textit{nonsingular (i.e. invertible)} \end{cases} ?$$

$$\frac{d}{dt}\left(\Phi(t)C\right) = \Phi'(t)C \quad \text{(universal product rule .... see last page of notes)}$$

Φ'

$$= (A\,\Phi)C$$
$$= A\,(\Phi C).$$

Also, $X(0) = \Phi(0)C$ is a product of invertible matrices, so is invertible as well. Thus $X(t) = \Phi(t)C$ is an FM.

(Notice this argument would not work if we had used $C\Phi(t)$ instead.)

$$(C\Phi)' = C\Phi' = CA\Phi$$
$$A(C\Phi) = AC\Phi \quad \cancel{=}$$

If $\Phi(t)$ is any FM for $\underline{x}' = A\,\underline{x}$ then $X(t) = \Phi(t)\Phi(0)^{-1}$ solves

$$\begin{cases} X'(t) = A\,X \\ X(0) = I \end{cases}$$

$$\begin{cases} \vec{x}_j' = A\vec{x}_j \\ \vec{x}_j(0) = \vec{e}_j \end{cases}$$

Notice that there is only one matrix solution to this IVP, since the j^{th} column $\underline{x}_j(t)$ is the (unique) solution to

$$\underline{x}'(t) = A\,\underline{x}$$
$$\underline{x}(0) = \underline{e}_j.$$

$$\Phi(t)\Phi(0)^{-1} := e^{tA}$$

Definition The unique FM that solves

$$\begin{cases} X'(t) = A\,X \\ X(0) = I \end{cases}$$

is called the matrix exponential, e^{tA} ...because:

(scalar case $\begin{cases} x' = ax \\ x(0) = 1 \end{cases}$ sol is scalar fcn $x(t) = e^{at}$)

This generalizes the scalar case. In fact, notice that if we wish to solve

$$IVP \begin{cases} \underline{x}'(t) = A\,\underline{x} \\ \underline{x}(0) = \underline{x}_0 \end{cases}$$

the solution is

$$\underline{x}(t) = e^{tA}\underline{x}_0,$$

$$\vec{x}(t) = \Phi(t)\Phi(0)^{-1}\vec{x}_0$$
$$= \Phi(t)\vec{x}_0 \quad \text{is } \Phi(0) = I$$

in analogy with Chapter 1.

1). $\vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\qquad \vec{x}(t) = e^{tA}\vec{x}_0$

$=$

Exercise 2) Continuing with our example, for the DE

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

with

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

and FM

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

compute e^{tA}. Check that the solution to the IVP in Exercise 1 is indeed $e^{tA}\underline{\boldsymbol{x}}_0$.

$$\Phi(0) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\Phi^{-1}(0) = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$e^{At} = \Phi(t)\,\Phi(0)^{-1} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} e^{-2t}+6e^{5t} & -2e^{-2t}+2e^{5t} \\ -3e^{-2t}+3e^{5t} & 6e^{-2t}+e^{5t} \end{bmatrix}$$

$$\text{(1)}\quad e^{At}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \left(\begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \left(\frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right)$$

$$@\, t=0:\ \frac{1}{7}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = I \checkmark$$

in (1) did mult from right to left

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> with(LinearAlgebra) :
  A := [4  2; 3  -1] :
  MatrixExponential(t·A);   #check work on previous page
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$$\begin{bmatrix} \frac{1}{7}e^{-2t} + \frac{6}{7}e^{5t} & \frac{2}{7}e^{5t} - \frac{2}{7}e^{-2t} \\ \frac{3}{7}e^{5t} - \frac{3}{7}e^{-2t} & \frac{6}{7}e^{-2t} + \frac{1}{7}e^{5t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \textbf{(8)}$$

>

But wait!
Didn't you like how we derived Euler's formula using Taylor series?
Here's an alternate way to think about e^{tA}:

For $A_{n \times n}$ consider the matrix series

$$e^{A} := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots + \frac{1}{k!}A^k + \ldots$$

Convergence: pick a large number M, so that each entry of A satisfies $|a_{ij}| \leq M$. Then

$$entry_{ij}(A^2) \leq nM^2$$
$$entry_{ij}(A^3) \leq n^2M^3 \ldots$$
$$entry_{ij}(A^k) \leq n^{k-1}M^k$$

$$1 + Mn + \frac{(Mn)^2}{2!} + \frac{(Mn)^3}{3!} + \ldots$$

so the matrix series converges absolutly in each entry (dominated by the Calc 2 series for the scalar e^{Mn}).

Then define

$$X(t) = e^{tA} := I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots + \frac{t^k}{k!}A^k + \ldots$$

Notice that for $X(t) = e^{tA}$ defined by the power series above, and assuming the true fact that we may differentiate the series term by term,

$$X'(\mathrm{t}) = 0 + A + \frac{2t}{2!}A^2 + \frac{3t^2}{3!}A^3 + \ldots \frac{kt^{k-1}}{k!}A^k + \ldots$$
$$= A + tA^2 + \frac{t^2}{2!}A^3 + \ldots \frac{t^{k-1}}{(k-1)!}A^k$$
$$= A\left(I + tA + \frac{t^2}{2!}A^2 + \ldots + \frac{t^{(k-1)}}{(k-1)!}A^{k-1} + \ldots\right)$$
$$= AX.$$

Also,

$$X(0) = I.$$

Thus, since there is only one matrix function that can satisfy

$$\begin{cases} X'(\mathrm{t}) = AX \\ X(0) = I \end{cases}$$

we deduce

<u>Theorem</u> The matrix exponential e^{tA} may be computed either of two ways:

$$e^{tA} = \Phi(t)\Phi(0)^{-1}$$
$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots + \frac{t^k}{k!}A^k + \ldots$$

Exercise 3 Let A be a diagonal matrix Λ,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \ldots & 0 \\ 0 & \lambda_2 & 0 \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Use the Taylor series definition and the FM definition to verify twice that

$$e^{t\Lambda} = \begin{bmatrix} e^{t\lambda_1} & 0 & \ldots & 0 \\ 0 & e^{t\lambda_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & e^{t\lambda_n} \end{bmatrix}$$

✓

Hint: products of diagonal matrices are diagonal, and the diagonal entries multiply, so

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \ldots & 0 \\ 0 & \lambda_2^k & 0 \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n^k \end{bmatrix}$$

$$e^{\Lambda t} = I + t\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ 0 & & \ddots \ \lambda_n^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2}{2!}\lambda_1^2 + \frac{t^3}{3!}\lambda_1^3 + \cdots & & 0 \\ & 1 + t\lambda_2 + \frac{t^2}{2!}\lambda_2^2 + \cdots & \\ 0 & & \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \ e^{\lambda_n t} \end{bmatrix}.$$

$$\vec{x}'(t) = \Lambda \vec{x}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{bmatrix}$$

$$\begin{aligned} x_1' &= \lambda_1 x_1 \\ x_2' &= \lambda_2 x_2 \\ &\vdots \\ x_n' &= \lambda_n x_n \end{aligned} \Rightarrow \begin{aligned} x_1 &= c_1 e^{\lambda_1 t} \\ x_2 &= c_2 e^{\lambda_2 t} \\ &\vdots \\ x_n &= c_n e^{\lambda_n t} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = c_1\begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{\lambda_2 t} \\ 0 \\ \vdots \end{bmatrix} \cdots + c_n\begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ e^{\lambda_n t} \end{bmatrix}$$

a FM would be $\Phi(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & & 0 \\ 0 & 0 & & \\ \vdots & \vdots & & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$

$\Phi(0) = I$ so $\Phi(t) = e^{tA}$.

Example How to recompute e^{tA} for

$$A := \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

using power series and Math 2270: The similarity matrix made of eigenvectors of A

$$S = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\begin{array}{cc} \vec{v} & \vec{w} \end{array}$$

$A\vec{v} = -2\vec{v}$

$A\vec{w} = 5\vec{w}$

yields

$$A\,S = S\,\Lambda:$$

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

so

$$A = S\Lambda S^{-1}.$$

$A^2 = S\Lambda \underbrace{S^{-1} S}\Lambda S^{-1} = S\Lambda^2 S^{-1}$

Thus $A^k = S\Lambda^k S^{-1}$ (telescoping product), so

$A^m = (S\Lambda \underbrace{S^{-1})(S}_{I}\Lambda \underbrace{S^{-1})\cdots(S}_{I}\Lambda S^{-1}) = S\Lambda^m S^{-1}$

$A^k = S\Lambda^k S^{-1}$

$$\begin{aligned} e^{tA} &= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots + \frac{t^k}{k!}A^k + \ldots \\ &= S\left(I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \ldots + \frac{t^k}{k!}\Lambda^k + \ldots\right)S^{-1} \\ &= S\,e^{t\Lambda}S^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{5t} \end{bmatrix}\frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} e^{-2t} & -2e^{-2t} \\ 3e^{5t} & e^{5t} \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix} \end{aligned}$$

• same.

which agrees with our original computation using the FM.

Three important properties of matrix exponentials:

- 1) $e^{[0]} = I$, where $[0]$ is the $n \times n$ zero matrix. (Why is this true?) power series def.

2) If $AB = BA$ then $e^{A+B} = e^A e^B = e^B e^A$ (but this identity is not generally true when A and B don't commute). (See homework.)

3) $\left(e^A\right)^{-1} = e^{-A}$. (Combine (1) and (2).)

tomorrow

$\vec{x}' = A\vec{x} + \vec{f}(t)$

$$e^{A-A} = e^0 = I$$

$$\underbrace{e^A}\,\underbrace{e^{-A}} = I$$

$$\left(e^A\right)^{-1} = e^{-A}$$

$$e^{A+B} = I + (A+B) + \boxed{\tfrac{1}{2!}(A+B)^2} + \cdots$$

$$e^A e^B = \left(I + A + \tfrac{A^2}{2!} + \cdots\right)\left(I + B + \tfrac{B^2}{2!} + \cdots\right)$$

$$= I + (A+B) + \left(\tfrac{A^2}{2} + AB + \tfrac{B^2}{2}\right) + \cdots$$

$$\tfrac{1}{2}(A+B)(A+B) = \tfrac{1}{2}\left(A^2 + AB + BA + B^2\right)$$

Using these properties there is a "straightforward" algorithm to compute e^{tA} even when A is not diagonalizable (and it doesn't require the use of chains studied in section 5.5). See Theorem 3 in section 5.6 We'll study more details on Monday, but here's an example:

<u>Exercise 4)</u> Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Find e^{tA} by writing $A = D + N$ where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and using $e^{tD + tN} = e^{tD} e^{tN}$. Hint: $N^3 = 0$ so the Taylor series for e^{tN} is very short.

Universal product rule for differentiation: Recall the 1-variable product rule for differentiation for a function of a single variable t, based on the limit definition of derivative. We'll just repeat that discussion, but this time for any product "$*$" that distributes over addition, for scalar, vector, or matrix functions. We also assume that for the product under consideration, scalar multiples s behave according to

$$s(f*g) = (sf)*g = f*(sg).$$

We don't assume that $f*g = g*f$ so must be careful in that regard. Here's how the proof goes:

$$D_t(f*g)(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(f(t+\Delta t)*g(t+\Delta t) - f(t)*g(t)\right).$$

We add and subtract a middle term, to help subsequent algebra:

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(f(t+\Delta t)*g(t+\Delta t) - f(t+\Delta t)*g(t) + f(t+\Delta t)*g(t) - f(t)*g(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(f(t+\Delta t)*g(t+\Delta t) - f(t+\Delta t)*g(t)\right) + \frac{1}{\Delta t}\left(f(t+\Delta t)*g(t) - f(t)*g(t)\right).$$

We assume that multiplication by $*$ distributes over addition:

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} f(t+\Delta t)*\left(g(t+\Delta t) - g(t)\right) + \frac{1}{\Delta t}\left(f(t+\Delta t) - f(t)\right)*g(t).$$

The sum rule for limits and rearranging the scalar factor let's us rearrange as follows:

$$= \lim_{\Delta t \to 0} f(t+\Delta t)*\left(\frac{1}{\Delta t}\right)\left(g(t+\Delta t) - g(t)\right) + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(f(t+\Delta t) - f(t)\right)*g(t).$$

Differentiable functions are continuous, so we take limits and get:

$$D_t(f*g)(t) = f(t)*g'(t) + f'(t)*g(t).$$

The proof above applies to

scalar function times scalar function (Calc I)
scalar function time vector function (Calc III)
dot product or cross product of two functions (Calc III)
scalar function times matrix function (our class)
matrix function times matrix or vector function (our class)

This proof does not apply to composition $f \circ g(t) := f(g(t))$, because composition does not generally distribute over addition, and this is why we have the chain rule for taking derivatives of composite functions.

$$\frac{d}{dt} A(t)B(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[A(t+\Delta t)B(t+\Delta t) - A(t)B(t)\right] \qquad A_{m\times n} B_{n\times p}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[A(t+\Delta t)B(t+\Delta t) - A(t+\Delta t)B(t) + A(t+\Delta t)B(t) - A(t)B(t)\right]$$

$$= \lim_{\Delta t \to 0} \cancel{\frac{1}{\Delta t}} A(t+\Delta t)\left(\frac{B(t+\Delta t) - B(t)}{\Delta t}\right) + \cancel{\frac{1}{\Delta t}}\left(\frac{A(t+\Delta t) - A(t)}{\Delta t}\right)B(t)$$

$$= A(t)B'(t) + A'(t)B(t)$$

$$= A'B + AB'$$

Monday:

- Matrix exponential HW postponed to after midterm
- midterm material thru 4 5.4 forced oscillations
- (this week: matrix exponentials & applications
- next week: 6: non-linear aut. sys of DE's.)
- Lab tomorrow: review: we'll go through parts of midterm from spring 2015

Math 2280-001 Week 11
Mar 27-29 (Exam 2 on Mar 31)

Mon Mar 27: Use last Friday's notes to discuss matrix exponentials

Wed Mar 29: 5.6-5.7 Matrix exponentials, linear systems, and variation of parameters for inhomogeneous systems.

Recall: For the first order system

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$

- $\Phi(t)$ is a fundamental matrix (FM) if its n columns are a basis for the solution space to the first order system above (i.e. $\Phi(t)$ is the Wronskian matrix for a basis to the solution space).

- $\Phi(t)$ is an FM if and only if $\Phi'(t) = A\,\Phi$ and $\Phi(0)$ is invertible.

- e^{tA} is the unique matrix solution to

$$X'(t) = A\,X$$
$$X(0) = I$$

and may be computed either of two ways:

$$e^{tA} = \Phi(t)\Phi(0)^{-1}$$

where $\Phi(t)$ is any other FM, or via the infinite series

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots + \frac{t^k}{k!}A^k + \ldots$$

Example 1: We showed that if

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \ldots & 0 \\ 0 & \lambda_2 & 0 \ldots & 0 \\ : & : & & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

then

$$e^{t\Lambda} = \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2}{2!}\lambda_1^2 + \ldots & 0 & 0 \ldots & 0 \\ 0 & 1 + t\lambda_2 + \frac{t^2}{2!}\lambda_2^2 + \ldots & 0 \ldots & 0 \\ : & : & & : \\ 0 & 0 & \ldots & 1 + t\lambda_n + \frac{t^2}{2!}\lambda_n^2 + \ldots \end{bmatrix}$$

$$=$$