Forced oscillations (still undamped):

$$M\underline{\boldsymbol{x}}''(t) = K\underline{\boldsymbol{x}} + \underline{\boldsymbol{F}}(t)$$
$$\Rightarrow \underline{\boldsymbol{x}}''(t) = A\underline{\boldsymbol{x}} + M^{-1}\underline{\boldsymbol{F}}(t)\ .$$

If the forcing is sinusoidal,

$$M\underline{\boldsymbol{x}}''(t) = K\underline{\boldsymbol{x}} + \cos(\omega t)\underline{\boldsymbol{G}}_0$$
$$\Rightarrow \underbrace{\underline{\boldsymbol{x}}''(t) - A\underline{\boldsymbol{x}}}_{L(\vec{x}(t))} = \underbrace{\cos(\omega t)\underline{\boldsymbol{F}}_0}$$

non-homog linear.

with $\underline{\boldsymbol{F}}_0 = M^{-1}\underline{\boldsymbol{G}}_0$.

From the fundamental theorem for linear transformations we know that the general solution to this inhomogeneous linear problem is of the form

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_P(t) + \underline{\boldsymbol{x}}_H(t)\ ,$$

and we've been discussing how to find the homogeneous solutions $\underline{\boldsymbol{x}}_H(t)$.

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\underline{\boldsymbol{x}}_P(t) = \cos(\omega t)\,\underline{\boldsymbol{c}}$$

$L(\cos(\omega t)\vec{c}) = \cos\omega t\,\vec{F}_0$
want

where the vector $\underline{\boldsymbol{c}}$ is what we need to find.

Exercise 2) Substitute the guess $\underline{\boldsymbol{x}}_P(t) = \cos(\omega t)\,\underline{\boldsymbol{c}}$ into the DE system

$$* \quad \underline{\boldsymbol{x}}''(t) = A\underline{\boldsymbol{x}} + \cos(\omega t)\underline{\boldsymbol{F}}_0$$

to find a matrix algebra formula for $\underline{\boldsymbol{c}} = \underline{\boldsymbol{c}}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

$\vec{x}_p(t) = (\cos\omega t)\vec{c}$.

LHS $\quad \vec{x}_p''(t) = \boxed{-\omega^2\cos\omega t\ \vec{c}}$ want equal

RHS $\quad A\vec{x}_p + \cos\omega t\,\vec{F}_0 = A\cos\omega t\,\vec{c} + \cos\omega t\,\vec{F}_0$

$\boxed{= \cos\omega t\,(A\vec{c} + \vec{F}_0)}$

need $\ -\omega^2\vec{c} = A\vec{c} + \vec{F}_0$

$-\vec{F}_0 = A\vec{c} + \omega^2\vec{c}$

$-\vec{F}_0 = (A + \omega^2 I)\vec{c}$

$(A + \omega^2 I)^{-1}(-\vec{F}_0) = \vec{c}$

Solution:

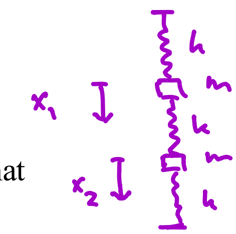
$$\boxed{\underline{\boldsymbol{c}}(\omega) = -\left(A + \omega^2 I\right)^{-1}\underline{\boldsymbol{F}}_0}\ .$$

Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue.

Exercise 3) Continuing with the configuration from Monday's notes, but now for an inhomogeneous forced problem, let $k = m$, and force the second mass sinusoidally:

$= \frac{N}{m} = \frac{kg}{s^2}$ $\quad$ kg.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



We know from previous work that the natural frequencies are $\omega_1 = 1$, $\omega_2 = \sqrt{3}$ and that

$$\underline{\boldsymbol{x}}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\, t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\underline{\boldsymbol{x}}_P(t)$, as on the preceding page. Notice that this steady periodic solution blows up as $\omega \to 1$ or $\omega \to \sqrt{3}$. (If we don't have time to work this by hand, we may skip directly to the technology check on the next page. But since we have quick formulas for inverses of 2 by 2 matrices, this is definitely a computation we could do by hand.)

$\lambda = -1, -3$

$\omega = 1, \sqrt{3}$

$E_{\lambda=-1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$E_{\lambda=-3} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

$\vec{c} = -(A + \omega^2 I)^{-1} \vec{F}_0$ $\qquad$ $x_p = \cos \omega t\, \vec{c}$

$\vec{c} = -\begin{bmatrix} -2+\omega^2 & 1 \\ 1 & -2+\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\vec{c} = \frac{-1}{(\omega^2-2)^2 - 1} \begin{bmatrix} -2+\omega^2 & -1 \\ -1 & -2+\omega^2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \frac{1}{(\omega^2-3)(\omega^2-1)} \begin{pmatrix} 3 \\ 3(2-\omega^2) \end{pmatrix}$

$x_p = \cos \omega t\, \vec{c}$

Solution: As long as $\omega \neq 1, \sqrt{3}$, the general solution $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t) \begin{bmatrix} \dfrac{3}{(\omega^2 - 1)(\omega^2 - 3)} \\ \dfrac{6 - 3\omega^2}{(\omega^2 - 1)(\omega^2 - 3)} \end{bmatrix} \checkmark$$

$$+ \; C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\, t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Interpretation as far as inferred practical resonance for slightly damped problems: If there was even a small amount of damping, the homogeneous solution would actually be transient (it would be exponentially decaying and oscillating - underdamped). There would still be a sinusoidal particular solution, which would have a formula close to our particular solution, the first term above, as long as $\omega \neq 1, \sqrt{3}$. (There would also be a relatively smaller $\sin(\omega t)\underline{\boldsymbol{d}}$ term as well.) So we can infer the practical resonance behavior for different ω values with slight damping, by looking at the size of the $\underline{\boldsymbol{c}}(\omega)$ term for the undamped problem....see next page for visualizations.

```
> restart :
> with(LinearAlgebra) :
> A := Matrix(2, 2, [-2, 1, 1, -2]) :
> F0 := Vector([0, 3]) :
> Iden := IdentityMatrix(2) :
```

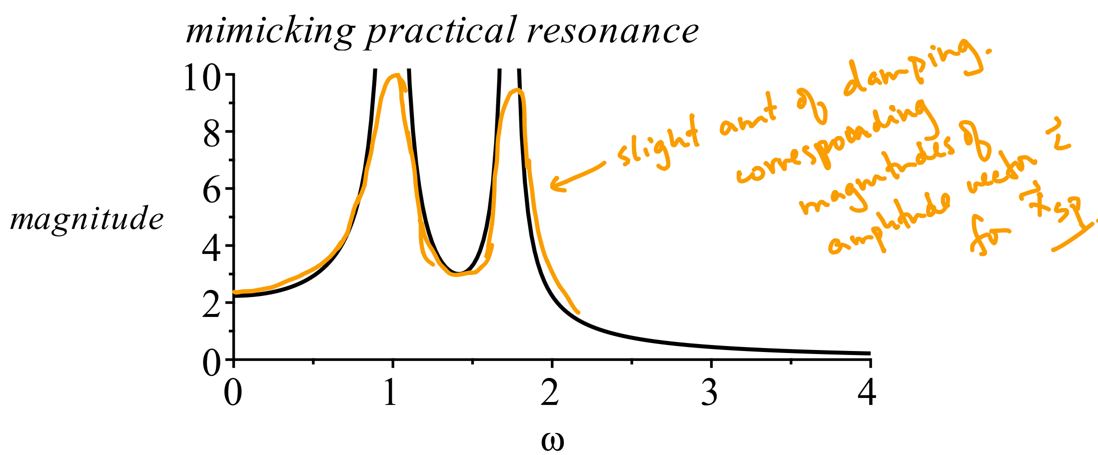
> $c := \omega \to \left(A + \omega^2 \cdot Iden\right)^{-1}.(-F0)$: # the formula we worked out by hand

> $c(\omega)$;

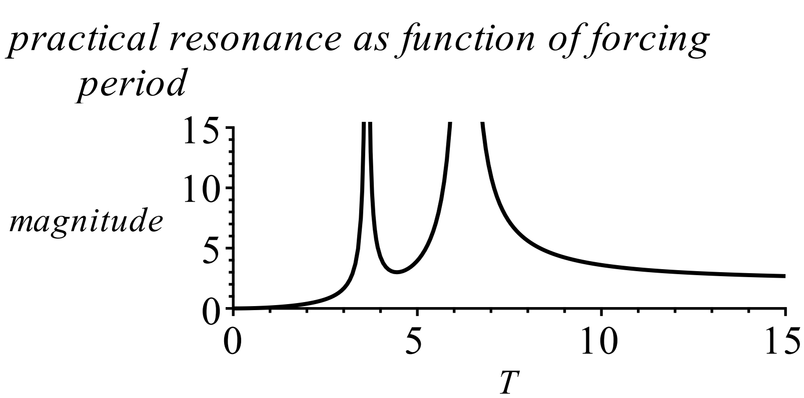
$$\begin{bmatrix} \dfrac{3}{\omega^4 - 4\,\omega^2 + 3} \\ -\dfrac{3\left(\omega^2 - 2\right)}{\omega^4 - 4\,\omega^2 + 3} \end{bmatrix} \qquad \textbf{(4)}$$

```
> with(plots) :
  with(LinearAlgebra) :
```

> plot(Norm(c(ω), 2), ω = 0 ..4, magnitude = 0 ..10, color = black, title = `mimicking practical resonance`);
> # Norm(c(ω),2) is the magnitude of the c(ω) vector



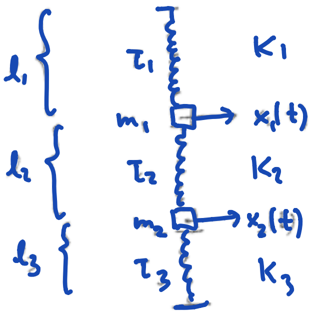
> plot$\left(Norm\left(c\left(\frac{2 \cdot Pi}{T}\right), 2\right)\right.$, T = 0 ..15, magnitude = 0 ..15, color = black, title = `practical resonance as function of forcing period`);



>

There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings:

- Transverse oscillations! (i.e. directions $\perp$ to the mass-spring configuration)



τ_1, τ_2, τ_3 are the **tensions** (pulling forces) of the stretched springs

By linearization, a good model would be

$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1+K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = K_2(x_1 - x_2) - K_3 x_2 = K_2 x_1 \quad -(K_2+K_3)x_2$$

where $K_1, K_2\ K_3$ are positive constants as before

→ but in general not the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2^{nd} order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & ~~fd~~ fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:

[force picture, e.g.

horiz force from top spring on mass 1

$$= -\tau_1 \sin\theta_1 = -\tau_1 \frac{x_1}{\sqrt{l_1^2+x_1^2}} \approx -\tau_1 \frac{x_1}{l_1} = -\frac{\tau_1}{l_1} x_1$$

So $K_1 = \frac{\tau_1}{l_1}$

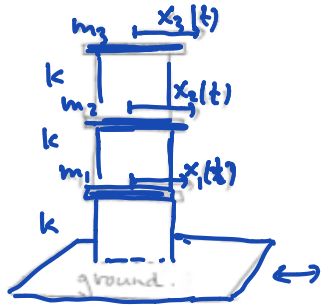
similarly, $K_2 = \frac{\tau_2}{l_2}$, $K_3 = \frac{\tau_3}{l_3}$

for our physics demo springs, equilibrium length ≈ 0, very Hookesian so $\tau \cong k\,l$; $\frac{\tau}{l} \approx k$, so actually almost recover same fundamental modes !!]

- An interesting shake-table demonstration:

http://www.youtube.com/watch?v=M_x2jOKAhZM

Below is a discussion of how to model the unforced "three-story" building shown shaking in the video above, from which we can see which modes will be excited. There is also a "two-story" building model in the video, and its matrix and eigendata follow. Here's a schematic of the three-story building:



$m_1 = m_2 = m_3 = m$

$$m_3 x_3'' = k(x_2 - x_3)$$
$$= -k x_3 + k x_2$$
$$m_2 x_2'' = +k(x_1 - x_2) + k(x_3 - x_2)$$
$$= k x_1 - 2k x_2 + k x_3$$
$$m_1 x_1'' = -k x_1 + k(x_2 - x_1)$$
$$m_1 x_1'' = -2k x_1 + k x_2$$

For the unforced (homogeneous) problem, the accelerations of the three massive floors (the top one is the roof) above ground and of mass m, are given by

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \frac{k}{m} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Note the -1 value in the last diagonal entry of the matrix. This is because $x_3(t)$ is measuring displacements for the top floor (roof), which has nothing above it. The "k" is just the linearization proportionality factor, and depends on the tension in the walls, and the height between floors, etc, as discussed on the previous page.

Here is eigendata for the unscaled matrix $\left(\frac{k}{m} = 1\right)$. For the scaled matrix you'd have the same eigenvectors, but the eigenvalues would all be multiplied by the scaling factor $\frac{k}{m}$ and the natural frequencies would all be scaled by $\sqrt{\frac{k}{m}}$. Symmetric matrices like ours (i.e matrix equals its transpose) are always diagonalizable with real eigenvalues and eigenvectors...and you can choose the eigenvectors to be mutually perpendicular. This is called the "Spectral Theorem for symmetric matrices" and is an important fact in many math and science applications...you can read about it here: http://en.wikipedia.org/wiki/Symmetric_matrix.) If we tell Maple that our matrix is symmetric it will not confuse us with unsimplified numbers and vectors that may look complex rather than real.

```
> with(LinearAlgebra) :
> A := Matrix(3, 3, [-2.0, 1, 0, 1, -2, 1, 0, 1, -1]);
      # I used at least one decimal value so Maple would evaluate in floating point
```

$$A := \begin{bmatrix} -2.0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \textbf{(5)}$$

```
> Digits := 5 :  # 5 digits should be fine, for our decimal approximations.
```

```
> eigendata := Eigenvectors(Matrix(A, shape=symmetric)) : # to take advantage of the
                 # spectral theorem
  lambdas := eigendata[1] :   #eigenvalues
  evectors := eigendata[2] :   #corresponding eigenvectors - for fundamental modes
  omegas := map(sqrt, -lambdas);  # natural angular frequencies
  f := x -> 2·evalf(Pi)/x :
  periods := map(f, omegas); #natural periods
  eigenvectors := map(evalf, evectors); #  get digits down to 5
```

($C_1 \cos(\omega_1 t - \alpha_1)\vec{v}_1$ slow "sloshing" mode.

$$\sqrt{-\lambda} = omegas := \begin{bmatrix} 1.8019 \\ 1.2470 \\ 0.44504 \end{bmatrix}$$

$$periods := \begin{bmatrix} 3.4870 \\ 5.0386 \\ 14.118 \end{bmatrix}$$

vectors of amplitudes

$$eigenvectors := \begin{bmatrix} -0.59101 & -0.73698 & 0.32799 \\ 0.73698 & -0.32799 & 0.59101 \\ -0.32799 & 0.59101 & 0.73698 \end{bmatrix} \quad \begin{matrix} \text{1st floor} \\ \text{2nd floor} \\ \text{3rd floor.} \end{matrix} \tag{6}$$

$\omega_3 = 1.80$ $\omega_2 = 1.25$ $\omega_1 = .445$

```
> B := Matrix(2, 2, [-2, 1.0, 1, -1]) :
  eigendata := Eigenvectors(Matrix(B, shape=symmetric)) : # to take advantage of the
                 # spectral theorem
  lambdas := eigendata[1] :   #eigenvalues
  evectors := eigendata[2] :   #corresponding eigenvectors - for fundamental modes
  omegas := map(sqrt, -lambdas);  # natural angular frequencies
  f := x -> 2·evalf(Pi)/x :
  periods := map(f, omegas); #natural periods
  eigenvectors := map(evalf, evectors); #  get digits down to 5
```

single-story $\omega = 1$.

$$omegas := \begin{bmatrix} 1.6180 \\ 0.61804 \end{bmatrix}$$

$$periods := \begin{bmatrix} 3.8834 \\ 10.166 \end{bmatrix}$$

$$eigenvectors := \begin{bmatrix} -0.85065 & -0.52573 \\ 0.52573 & -0.85065 \end{bmatrix} \tag{7}$$

```
>
```

$\omega_2 = 1.62$ $\omega_1 = .618$

Exercise 4) Interpret the data above, in terms of the natural modes for the shaking building . In the youtube video the first mode to appear is the slow and dangerous "sloshing mode", where all three floors oscillate in phase, with amplitude ratios 33 : 59 : 74 from the first to the third floor. What's the second mode that gets excited? The third mode? (They don't show the third mode in the video.)

Remark) All of the ideas we've discussed in section 5.4 also apply to molecular vibrations. The eigendata in these cases is related to the "spectrum" of light frequencies that correspond to the natural fundamental modes for molecular vibrations.

Friday : finish mass-spring systems
quiz at end of class

Math 2280-001
Fri Mar 24

Monday:
Tuesday: review session : go through old midterm.
Wed : continue w § 5.7

5.6 Matrix exponentials and linear systems: The analogy between first order systems of linear differential equations (Chapter 5) and scalar linear differential equations (Chapter 1) is much stronger than you may have expected. This will become especially clear on Monday, when we study section 5.7.

<u>Definition</u> Consider the linear system of differential equations for $x(t)$:

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

where $A_{n \times n}$ is a constant matrix as usual. If $\{\underline{\boldsymbol{x}}_1(t), \underline{\boldsymbol{x}}_2(t), \ldots \underline{\boldsymbol{x}}_n(t)\}$ is a basis for the solution space to this system, then the matrix having these solutions as columns,

$$\Phi(t) := \left[\underline{\boldsymbol{x}}_1(t) \middle| \underline{\boldsymbol{x}}_2(t) \middle| \ldots \middle| \underline{\boldsymbol{x}}_n(t)\right]$$

is called a Fundamental Matrix (FM) to this system of differential equations. Notice that this equivalent to saying that $X(t) = \Phi(t)$ solves

$$\begin{cases} X'(\mathrm{t}) = A\,X \\ X(0) \quad\quad \textit{nonsingular (i.e. invertible)} \end{cases}$$

(just look column by column). <u>Notice that a FM is just the Wronskian matrix for a solution space basis.</u>

<u>Example 1</u> page 351

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda+2)(\lambda-5)$$

$\lambda = -2$:

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \Rightarrow \underline{\boldsymbol{v}} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\lambda = 5$:

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right] \Rightarrow \underline{\boldsymbol{v}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

general solution

$$\underline{\boldsymbol{x}}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Possible FM:

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

general solution:

$$\Phi(t)\underline{\boldsymbol{c}} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$