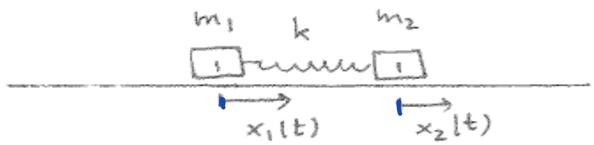
Exercise 4) Consider a train with two cars connected by a spring:



$\cancel{m_1} x_1''(t) = \frac{k}{m_1}(x_2 - x_1)$

$\cancel{m_2} x_2'' = -\frac{k}{m_2}(x_2 - x_1)$

4a) Derive the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero)

4b). Find the eigenvalues and eigenvectors. Then find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{\boldsymbol{v}}$ verify that you get two solutions

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{v}} \text{ and } \underline{\boldsymbol{x}}(t) = t\,\underline{\boldsymbol{v}},$$

rather than the expected $\cos(\omega t)\underline{\boldsymbol{v}}$, $\sin(\omega t)\underline{\boldsymbol{v}}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems.

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \underbrace{\begin{bmatrix} -d_1 & d_1 \\ d_2 & -d_2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -d_1-\lambda & d_1 \\ d_2 & -d_2-\lambda \end{vmatrix}$$

$$= (d_1+\lambda)(d_2+\lambda) - d_1 d_2$$

$$= \lambda^2 + (d_1+d_2)\lambda$$

$$= \lambda(\lambda + (d_1+d_2)).$$

$$\lambda = 0, \ -(d_1+d_2)$$

$$E_{\lambda=0}: \quad \left[\begin{array}{cc|c} -d_1 & d_1 & 0 \\ d_2 & -d_2 & 0 \\ \hline -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, soltns $(c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$E_{\lambda=-(d_1+d_2)} \quad \left[\begin{array}{cc|c} d_2 & d_1 & 0 \\ d_2 & d_1 & 0 \end{array}\right]$$

$\omega = \sqrt{d_1+d_2}$

$\vec{v} = \begin{bmatrix} d_1 \\ -d_2 \end{bmatrix}$ soltns $(c_3 \cos\omega t + c_4 \sin\omega t)\begin{bmatrix} d_1 \\ -d_2 \end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{(c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{"mode 1"}} + \underbrace{(c_3 \cos\omega t + c_4 \sin\omega t)\begin{bmatrix} d_1 \\ -d_2 \end{bmatrix}}_{\text{"mode 2"}}$$

"mode 1": horizontal transl. & const. veloc. motion.

"mode 2": oscillating out of phase, with amplitude ratio $d_1 : d_2$

Math 2280-001
Wed Mar 22

Today. Wed.
+ overview ✓
+ train example Monday ✓
+ experiment

5.4 Mass-spring systems and forced oscillation non-homogeneous problems.

• Finish Monday's notes if necessary, about unforced, undamped oscillations in multi mass-spring configurations. As a check of your understanding between first order systems and second order conservative mass-spring systems, see if you can answer the exercise below. Then proceed to experiment and forced oscillations on following pages.

Summary exercise: Here are two systems of differential equations, and the eigendata is as shown. The first order system could arise from an input-output model, and the second one could arise from an undamped two mass, three spring model. Write down the general solution to each system.

1a)

$$\begin{bmatrix} x_1{}' \\ x_2{}' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

input-output model

1b)

$$\begin{bmatrix} x_1{}'' \\ x_2{}'' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2 mass, 3 spring model

eigendata: For the matrix

$$A = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

for the eigenvalue $\underbrace{\lambda = -5, \underline{\boldsymbol{v}} = [-2, 1]^T}$, is an eigenvector; for the eigenvalue $\underbrace{\lambda = -1, \underline{\boldsymbol{v}} = [2, 1]^T}$ is an eigenvector

1a) $$c_1 e^{-5t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

soltn space 2-dim'l

1b) $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left(c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t\right)\begin{bmatrix} -2 \\ 1 \end{bmatrix} + \left(c_3 \cos t + c_4 \sin t\right)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda = -5$
$\omega = \sqrt{5} \quad (= \sqrt{-\lambda})$

$\lambda = -1$
$\omega = 1$

soltn space 4-dim'l
2 fdl' modes

Big picture (when A is diagonalizable) $\quad S^{-1}AS = \Lambda$

Λ = diagonal matrix w eigenvalues of A down the diagonal

same as

$$AS = S\Lambda$$

$$A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix}$$

S = matrix of eigenvector basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$)

1^{st} order:

$$\vec{x}' = A\vec{x}$$

$$S^{-1}\vec{x}' = S^{-1}A\vec{x}$$

Let $\vec{x} = S\vec{u}$

$$\vec{x}(t) = S\vec{u}(t)$$

$$\Rightarrow (\vec{x}'(t) = S\vec{u}'(t))$$

$$S^{-1}S\vec{u}'(t) = S^{-1}AS\vec{u}(t)$$

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \vec{u}'(t) = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

$$\begin{aligned} u_1' &= \lambda_1 u_1 \\ u_2' &= \lambda_2 u_2 \\ &\vdots \\ u_n' &= \lambda_n u_n \end{aligned} \quad \Rightarrow \quad \begin{aligned} u_1(t) &= c_1 e^{\lambda_1 t} \\ u_2(t) &= c_2 e^{\lambda_2 t} \\ &\vdots \\ u_n(t) &= c_n e^{\lambda_n t} \end{aligned}$$

so $\vec{x}(t) = S\vec{u}(t)$

$$\vec{x}(t) = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \end{bmatrix}$$

$$\boxed{\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots + c_n e^{\lambda_n t}\vec{v}_n}$$

2^{nd} order:

$$\vec{x}'' = A\vec{x}$$

$$S^{-1}\vec{x}'' = S^{-1}A\vec{x}$$

Let $\vec{x}(t) = S\vec{u}(t)$

$$\vec{x}''(t) = S\vec{u}''(t)$$

$$S^{-1}S\vec{u}'' = S^{-1}AS\vec{u}$$

$$\vec{u}'' = \Lambda\vec{u} \qquad u_j'' = \lambda_j u_j \qquad j = 1, 2, \ldots n$$

$\lambda_j < 0$, write $\lambda_j = -\omega_j^2$

$$u_j'' + \omega_j^2 u_j = 0$$

$$u_j(t) = A_j\cos(\omega_j t) + B_j\sin(\omega_j t)$$

if $\lambda_j = 0$, $u_j'' = 0$, $u_j(t) = c_1 + c_2 t$

(if $\lambda_j = 0$ corresp. soltns are $(c_1 + c_2 t)\vec{v}_j$)

$$\boxed{\begin{aligned} \vec{x}(t) &= S\vec{u} \\ &= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} A_1\cos\omega_1 t + B_1\sin\omega_1 t \\ A_2\cos\omega_2 t + B_2\sin\omega_2 t \\ \vdots \end{bmatrix} \end{aligned}}$$

The two mass, three spring system....Experiment!

Data: Each mass is 50 grams. Each spring mass is 10 grams. (Remember, and this is a defect, our model assumes massless springs.) The springs are "identical", and a mass of 50 grams stretches the spring 15.6 centimeters. (We should recheck this since it's old data; we should also test the spring's "Hookesiness"). With the old numbers we get Hooke's constant

```
> Digits := 4:
> solve(k·.156 = .05·9.806, k)
```

k 3.143 N/m. **(1)**

$mg = kx$

$(.05\text{ kg})\,g = k\,(.156)$

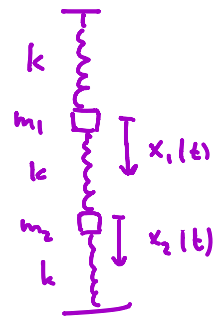
$\frac{(.05)(9.806)}{.156} = k$

```
>
```

Here's Maple confirmation for some of our work yesterday:

```
> with(LinearAlgebra):
  A := Matrix(2, 2, [-2·k/m, k/m, k/m, -2·k/m]);
  Eigenvectors(A);
```

$$A := \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \quad *$$



$$\begin{array}{c} \lambda_2 \\ \lambda_1 \end{array} \begin{bmatrix} -\frac{3k}{m} \\ -\frac{k}{m} \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \vec{v}_2,\ \vec{v}_1$$

$\cancel{m}\, x_1'' = -\frac{k}{m} x_1 + \frac{k}{m}(x_2 - x_1) = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$ **(2)**

$\cancel{m}\, x_2'' = -\frac{k}{m}(x_2 - x_1) - \frac{k}{m}x_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$

Predict the two natural periods from the model and our experimental value of k, m. Then make the system vibrate in each mode individually and compare your prediction to the actual periods of these two fundamental modes.

$\lambda_1 = -\frac{k}{m}$, $\omega_1 = \sqrt{\frac{k}{m}}$ $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -\frac{3k}{m}$, $\omega_2 = \sqrt{\frac{3k}{m}}$ $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$k = 3.143$ N/m

$m = .05$ kg

$\omega_1 = 7.928$

$f_1 = \frac{\omega_1}{2\pi}$

$T_1 = \frac{2\pi}{\omega_1} = \boxed{.792 \text{ sec/cycle}}$

$\omega_2 = \sqrt{3}\,\omega_1$

$T_2 = \frac{1}{\sqrt{3}} T_1 = \boxed{.458}$ sec/cycle

20 cycles = 17.18 sec

17.25

~~16.9~~

17.13

~~17.5~~

$\boxed{\frac{17.2}{20} = .86 \text{ sec/cycle}}$

20 cycles

9.2 sec

9.33

9.44

9.1

$\frac{9.27}{20} = \boxed{.464}$

ANSWER: If you do the model correctly and my office data is close to our class data, you will come up with theoretical natural periods of close to .46 and .79 seconds. I predict that the actual natural periods are a little longer, especially for the slow mode. (In my office experiment I got periods of 0.482 and 0.855 seconds.) What happened?

EXPLANATION: The springs actually have mass, equal to 10 grams each. This is almost on the same order of magnitude as the yellow masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more-complicated situation, using the ideas of total energy discussed in section 3.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator http://www.math.utah.edu/~korevaar/2280spring15/feb25.pdf, assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an *A*-matrix the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{6\,k}{6\,m + 5\,m_s}$$

$$\lambda_2 = -\frac{6\,k}{2\,m + m_s}.$$

(Hints: the "M" matrix is not diagonal but the "K" matrix is the same.)

If you use these values, then you get period predictions

```
> m := .05;
  ms := .010;
  k := 3.143;
  Omega1 := sqrt(6*k/(6*m + 5*ms));
  Omega2 := sqrt(6*k/(2*m + ms));
  T1 := evalf(2*Pi/Omega1);
  T2 := evalf(2*Pi/Omega2);
```

$m := 0.05$

$ms := 0.010$

$k := 3.143$

$\Omega 1 := 7.340$

$\Omega 2 := 13.09$

$\boxed{T1 := 0.8559 \quad T2 := 0.4801}$ .86 sec/cycle !!
.464 sec/cycle (3)

of .856 and .480 seconds per cycle. Is that closer?