

Friday: Friday's notes
 → 1st: lab tank problem.

5.3 phase diagrams for two linear systems of first order differential equations

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{x}' = A\vec{x}$$

Our goal is to understand how the (tangent vector field) phase portraits and solution curve trajectories are shaped by the eigendata of the matrix A . This discussion will be helpful in Chapter 6, when we discuss autonomous non-linear first order systems of differential equations, equilibrium points, and linearization near equilibrium points.

We will consider the cases of real eigenvalues and complex eigenvalues separately.

Real eigenvalues If the matrix $A_{2 \times 2}$ is diagonalizable, i.e. if there exists a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of \mathbb{R}^2 consisting of eigenvectors of A , then let λ_1, λ_2 be the corresponding eigenvalues (which may or may not be distinct).

- In this case, the general solution to the system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2$$

- And, for each $\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$ the value of the tangent field at \mathbf{x} is

$$A\mathbf{x} = A(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) = c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2.$$

(The text discusses the case of non-diagonalizable A . This can only happen if $\det(A - \lambda I) = (\lambda - \lambda_1)^2$, but the dimension of the $\lambda = \lambda_1$ eigenspace is only one.)

Exercise 1) This is an example of what happens when A has two real eigenvalues of opposite sign.

Consider the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Find the eigendata for A , and the general solution to the first order system of DE's.

b) (On the next page) use just the eigendata to sketch the tangent vector field (on the first plot). Begin by sketching the two eigenspaces.

c) (On the next page) use just the general solutions to the DE system to sketch representative solution curves (on the second plot).

Your answers to b,c should be consistent.

$$a) \begin{vmatrix} -3-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = (-3-\lambda)(4-\lambda) + 6 = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2) = 0$$

$$\lambda = 3, -2.$$

$$E_{\lambda=3}: \begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \\ \hline -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

$$R_{1/2} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{soln: } e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$E_{\lambda=-2}: \begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \\ \hline \end{array}$$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{soln: } e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$b) A(c_1 \vec{v}_1 + c_2 \vec{v}_2) = c_1 A \vec{v}_1 + c_2 A \vec{v}_2 \\ = c_1 3 \vec{v}_1 + c_2 (-2 \vec{v}_2)$$

$$A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A(c \begin{bmatrix} 2 \\ 1 \end{bmatrix}) = c A \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

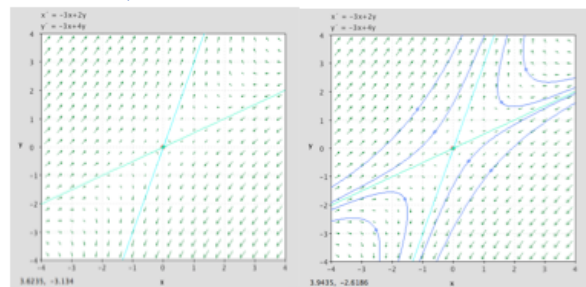
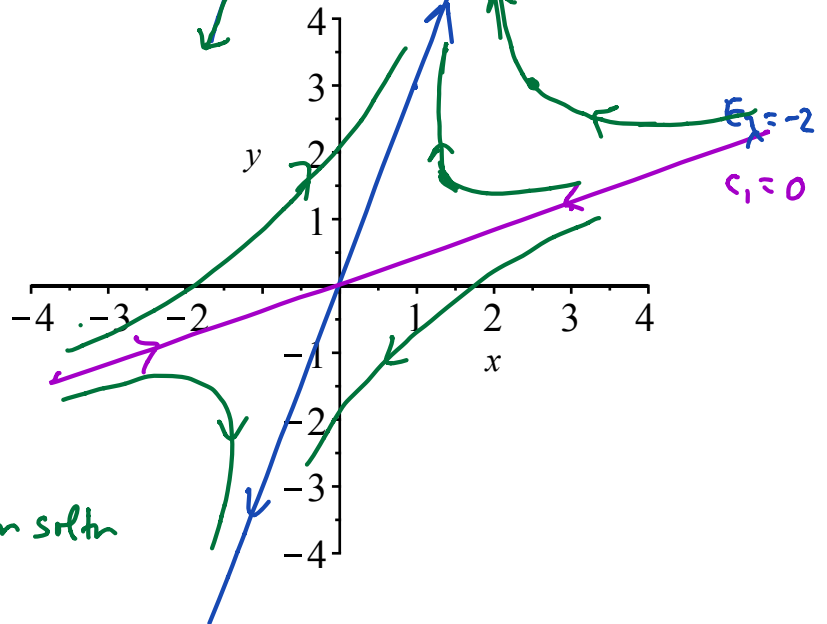
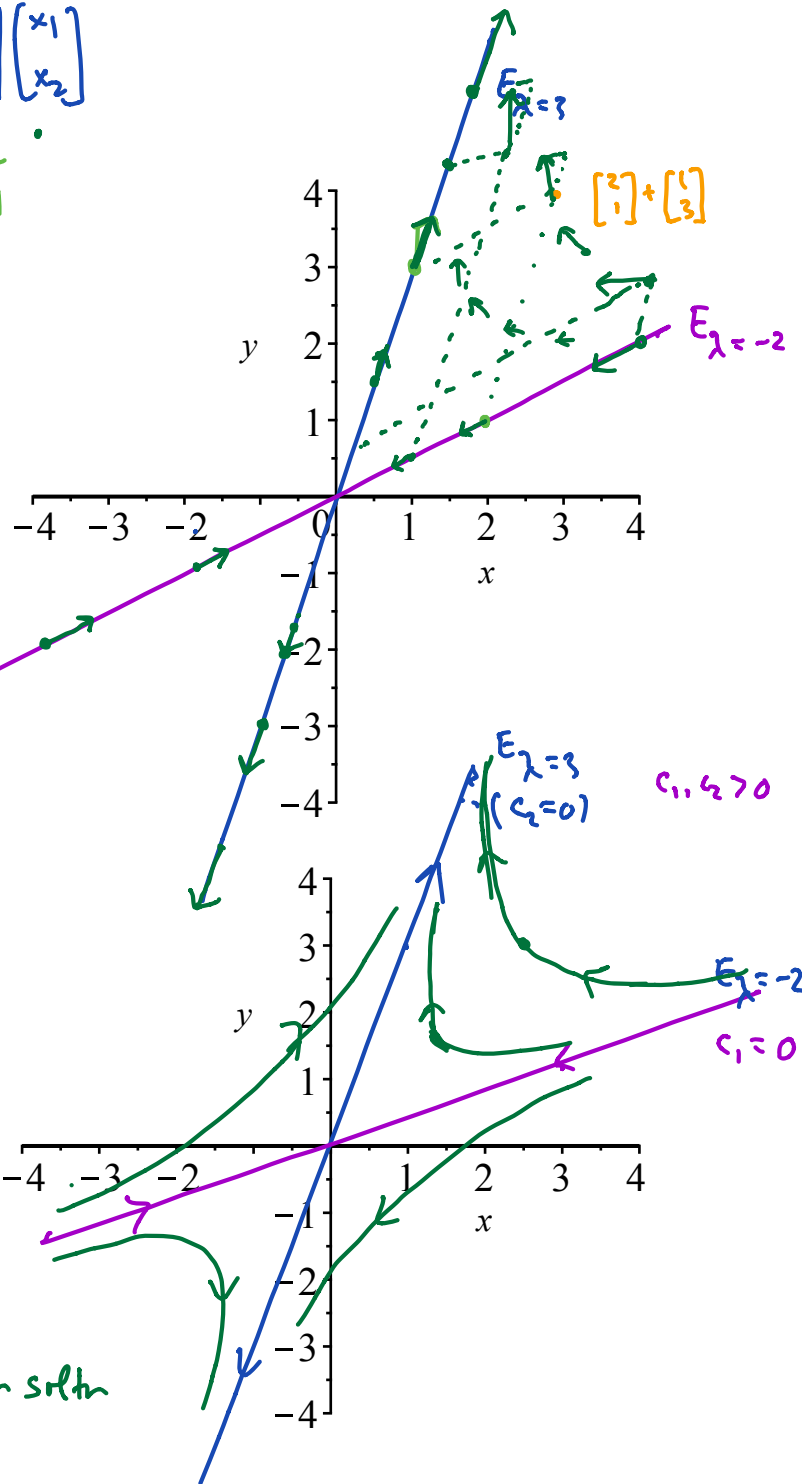
$$\cdot A(c \vec{v}) = c A(\vec{v})$$

tridians!

$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_1 < 0 < \lambda_2$$

origin is called
saddle pt. equilibrium soltn



Exercise 2) This is an example of what happens when A has two real eigenvalues of the same sign. Consider the system

$$\begin{bmatrix} x_1' (t) \\ x_2' (t) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- a) Find the eigendata for A , and the general solution to the first order system of DE's.
 b) Use the eigendata and the general solutions to construct a phase plane portrait of typical solution curves.

a) $\begin{vmatrix} s-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (s-\lambda)(4-\lambda)-2 = \lambda^2-9\lambda+18 = (\lambda-6)(\lambda-3)$

$$E_{\lambda=3} \quad \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\text{soln } e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

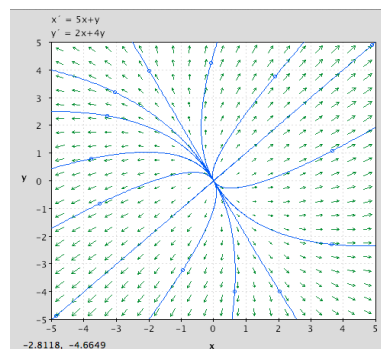
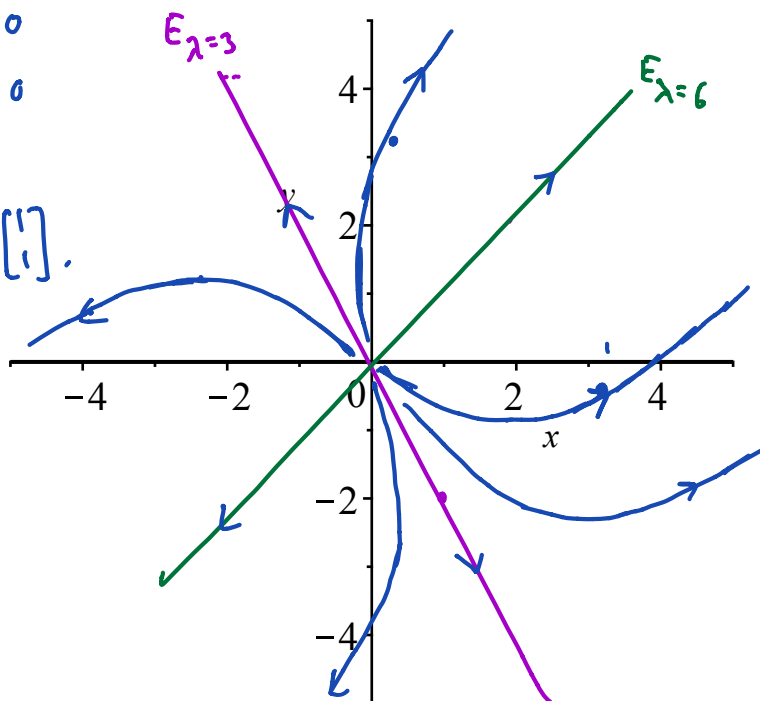
$$E_{\lambda=6} \quad \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{soln } e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_1, \lambda_2 > 0$
 (unstable) node.



Theorem: Time reversal: If $x(t)$ solves

$$x' = Ax$$

then $z(t) := x(-t)$ solves

$$z' = (-A)z$$

proof: by the chain rule,

$$z'(t) = x'(-t) \cdot (-1) = -x'(-t) = -Ax(-t) = -Az.$$

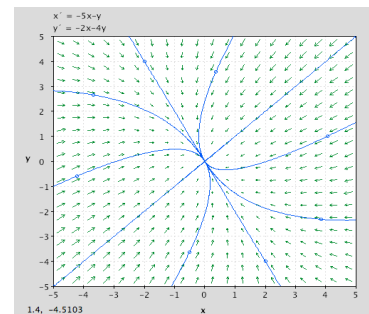
Exercise 3)

a) Let A be a square matrix, and let c be a scalar. How are the eigenvalues and eigenspaces of cA related to those of A ?

b) Describe how the eigendata of the matrix in the system below, is related to that of the (opposite) matrix in the previous exercise. Also describe how the phase portraits are related.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\lambda_1, \lambda_2 < 0$$



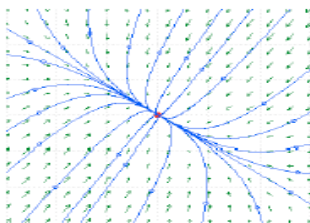
summary: In case the matrix $A_{2 \times 2}$ is diagonalizable with real number eigenvalues, the first order system of DE's

$$\mathbf{x}'(t) = A \mathbf{x}$$

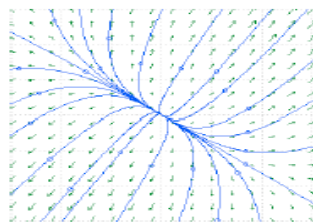
has general solution

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

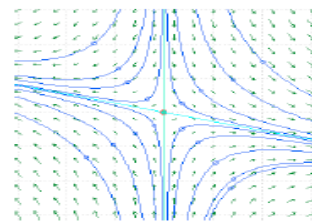
If each eigenvalue is non-zero, the three possibilities are:



nodal sink
 $\lambda_1, \lambda_2 < 0$



nodal source
 $\lambda_1, \lambda_2 > 0$



saddle point
 $\lambda_1 < 0 < \lambda_2$

complex eigenvalues: Consider the first order system

$$\mathbf{x}'(t) = A \mathbf{x}$$

Let $A_{2 \times 2}$ have complex eigenvalues $\lambda = p \pm q i$. For $\lambda = p + q i$ let the eigenvector be $\mathbf{v} = \mathbf{a} + \mathbf{b} i$.

Then we know that we can use the complex solution $e^{\lambda t} \mathbf{v}$ to extract two real vector-valued solutions, by taking the real and imaginary parts of the complex solution

$$\begin{aligned} \mathbf{z}(t) &= e^{\lambda t} \mathbf{v} = e^{(p+qi)t} (\mathbf{a} + \mathbf{b} i) \\ &= e^{p t} (\cos(q t) + i \sin(q t)) (\mathbf{a} + \mathbf{b} i) \\ &= [e^{p t} \cos(q t) \mathbf{a} - e^{p t} \sin(q t) \mathbf{b}] \\ &\quad + i [e^{p t} \sin(q t) \mathbf{a} + e^{p t} \cos(q t) \mathbf{b}] \end{aligned}$$

$$\begin{aligned} \bar{\mathbf{z}}' &= A \bar{\mathbf{z}} \\ \bar{\mathbf{x}}' + i \bar{\mathbf{y}}' &= A \mathbf{x} + i A \mathbf{y} \end{aligned}$$

Thus, the general real solution is a linear combination of the real and imaginary parts of the solution above:

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{p t} [\cos(q t) \mathbf{a} - \sin(q t) \mathbf{b}] \\ &\quad + c_2 e^{p t} [\sin(q t) \mathbf{a} + \cos(q t) \mathbf{b}] \end{aligned} = e^{p t} \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} \begin{bmatrix} c_1 \cos q t + c_2 \sin q t \\ -c_1 \sin q t + c_2 \cos q t \end{bmatrix}$$

We can rewrite $\mathbf{x}(t)$ as

$$\mathbf{x}(t) = e^{p t} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos(q t) & \sin(q t) \\ -\sin(q t) & \cos(q t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Breaking that expression down from right to left, what we have is:

- parametric circle of radius $\sqrt{c_1^2 + c_2^2}$, with angular velocity $\omega = -q$:

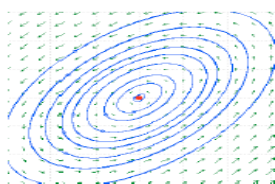
$$\begin{bmatrix} \cos(q t) & \sin(q t) \\ -\sin(q t) & \cos(q t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- transformed into a parametric ellipse by a matrix transformation of \mathbb{R}^2 :

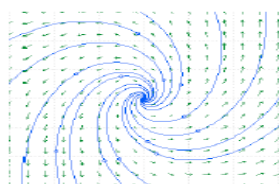
$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos(q t) & \sin(q t) \\ -\sin(q t) & \cos(q t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- possibly transformed into a shrinking or growing spiral by the scaling factor $e^{p t}$, depending on whether $p < 0$ or $p > 0$. If $p = 0$, curve remains an ellipse.

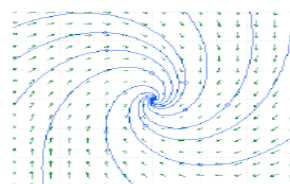
Thus $\mathbf{x}(t)$ traces out a stable spiral ("spiral sink") if $p < 0$, and unstable spiral ("spiral source") if $p > 0$, and an ellipse ("stable center") if $p = 0$:



center
 $\text{Re}(\lambda) = 0$



spiral source
 $\text{Re}(\lambda) > 0$



spiral sink
 $\text{Re}(\lambda) < 0$

Exercise 4) Do the eigendata analysis, find the general solution, and use tangent vectors just along the two axes to sketch typical solution curve trajectories, for this system from your homework due today:

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} &= \lambda(\lambda+2)+5 \\ &= \lambda^2+2\lambda+5 \\ &= (\lambda+1)^2+4=0 \\ (\lambda+1)^2 &= -4 \\ \lambda+1 &= \pm 2i \\ \lambda &= -1 \pm 2i \end{aligned}$$

