<u>Damped forced oscillations</u> $(c > 0)$ for $x(t)$:

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_P(t)$:

$$\begin{aligned}
k\,&\left[x_P = A\cos(\omega t) + B\sin(\omega t)\right] \\
+\,c\,&\left[x_P' = -A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t)\right] \\
+\,m\,&\left[x_P'' = -A\,\omega^2\cos(\omega t) - B\,\omega^2\sin(\omega t)\right] .
\end{aligned}$$

$$\begin{aligned}
L(x_P) &= \cos(\omega t)\left(k\,A + c\,B\,\omega - m\,A\,\omega^2\right) \\
&\quad + \sin(\omega t)\left(k\,B - c\,A\,\omega - m\,B\,\omega^2\right) .
\end{aligned}$$

want $= \cos \omega t\ (F_0)$ $+ \sin \omega t\ (0)$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\,\omega^2 & c\,\omega \\ -c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 & -c\,\omega \\ c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 \\ c\,\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} . \quad \text{(Check!)}$$

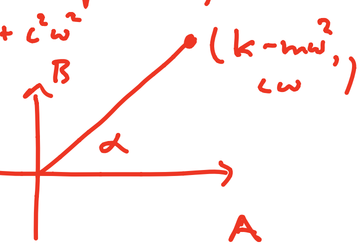
$= \dfrac{F_0}{(k-m\omega^2)^2 + c^2\omega^2} \sqrt{(k-m\omega^2)^2 + c^2\omega^2}$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}$$

$= \dfrac{m(\omega_0^2 - \omega^2)}{\text{denom}}$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} .$$

$k - m\omega^2$
$= m\left(\frac{k}{m} - \omega^2\right)$
$= m\left(\omega_0^2 - \omega^2\right)$



And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t} C_1 \cos\left(\omega_1 t - \alpha_1\right)$.
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t}\left(c_1 t + c_2\right)$.
- over-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

<u>Important to note:</u>

- The amplitude C in x_{sp} can be quite large relative to $\dfrac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is <u>practical resonance</u>.
- The phase angle α is always in the first or second quadrant.

Exercise 4) (a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice that our math on the previous page exactly predicts when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient c! Namely, for c small, when $\boxed{\omega^2 << \omega_0^2}$ we have α near zero (in phase) for x_{sp}, because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx 1$; when $\omega^2 >> \omega_0^2$ we have α near π (out of phase), because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx -1$; for $\omega \approx \omega_0$, α is near $\frac{\pi}{2}$, because $\sin(\alpha) \approx 1$, $\cos(\alpha) \approx 0$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

Exercise 5) Solve the IVP for $x(t)$:

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0.$$

Solution:

$$x(t) = \sqrt{41}\cos(4t - \alpha) + \sqrt{10}\, e^{-t}\cos(5t - \beta)$$
$$\alpha = \arctan(0.8),\ \beta = \arctan(-3).$$

$x = x_p + x_H$

$\downarrow$

$x'' + 2x' + 26x = 0$

$p(r) = r^2 + 2r + 26$

$= (r+1)^2 + 25 = 0$

$(r+1)^2 = -25$

$r + 1 = \pm 5i$

$r = -1 \pm 5i$

$x_H(t) = c_1 e^{-t}\cos 5t + c_2 e^{-t}\sin 5t$

§3.4

$+26\,(x_p = A\cos 4t + B\sin 4t)$

$+2\,(x_p' = -4A\sin 4t + 4B\cos 4t)$

$+1\,(x_p'' = -16A\cos 4t - 16B\sin 4t)$

want

$L(x_p) = \cos 4t\,(26A + 8B - 16A) + \sin 4t\,(26B - 8A - 16B) = 82\cos 4t + 0\sin 4t$

$10A + 8B = 82$

$-8A + 10B = 0$

$5A + 4B = 41$

$-4A + 5B = 0$

$$\begin{bmatrix} 5 & 4 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 41 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\cancel{41}}\begin{bmatrix} 5 & -4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} \cancel{41} \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$\boxed{x_p = 5\cos 4t + 4\sin 4t}$

$\boxed{x = x_p + x_H = 5\cos 4t + 4\sin 4t + c_1 e^{-t}\cos 5t + c_2 e^{-t}\sin 5t}$

```
> with(DEtools) :
> dsolve({x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});
```

$$x(t) = \underbrace{-3\,e^{-t}\sin(5t) + e^{-t}\cos(5t)}_{x_H} + \underbrace{5\cos(4t) + 4\sin(4t)}_{x_p} \quad (4)$$

Practical resonance: The steady periodic amplitude C for damped forced oscillations (two pages back) is

$$\boxed{C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}} \,.$$

Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$.

(Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in ω^2 it will have at most one minimum in the variable ω^2, so $C(\omega)$ will have at most one maximum for non-negative ω. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 6a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c:

$$\boxed{x'' + c\,x' + 26\,x = 82\cos(\omega\,t)}.$$

6b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

$x_{sp} = C(\omega)\cos(\omega t - \alpha)$

$$C(\omega) = \frac{82}{\sqrt{(26-\omega^2)^2 + 4\omega^2}}$$

shortcut: minimize

$$D_\omega\left[(26-\omega^2)^2 + 4\omega^2\right] = 2(26-\omega^2)(-2\cancel{\omega}) + 8\cancel{\omega} = 0$$

$$-4(26-\omega^2) + 8 = 0$$

$$8 = 4(26-\omega^2)$$

$$2 = 26-\omega^2$$

$$\omega^2 = 24 \Rightarrow \boxed{\omega = \sqrt{24}}$$

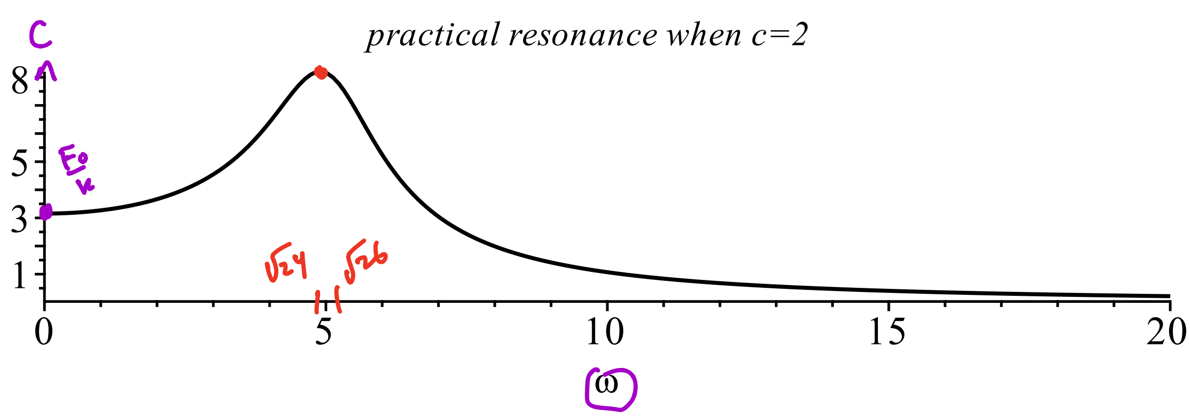
ω_0 for no damping is $\sqrt{26}$

amplitude of steady periodic soltn.

> $with(plots):$

> $C := (\omega, c) \to \dfrac{82}{\sqrt{\left(26-\omega^2\right)^2 + c^2\cdot\omega^2}}:$

> $plot(C(\omega, 2), \omega = 0..20, color = black, title = \text{`practical resonance when c=2`});$



>

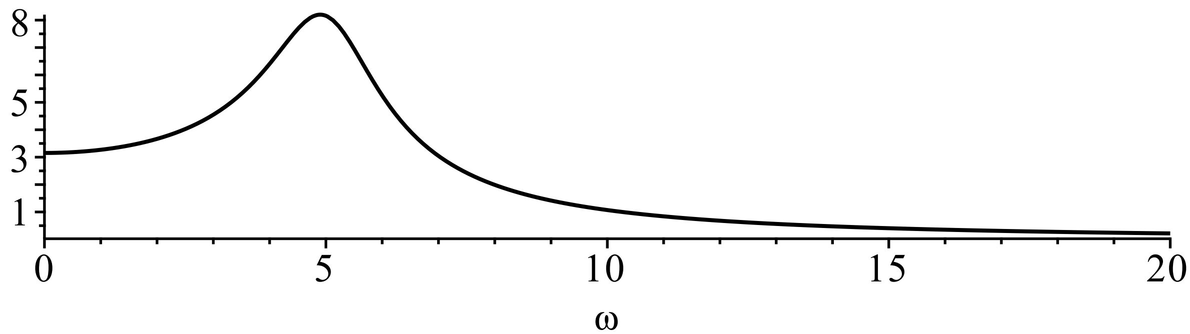
> *with*(*plots*) :

> $C := (\omega, c) \rightarrow \dfrac{82}{\sqrt{(26-\omega^2)^2 + c^2 \cdot \omega^2}}$:

c=2

> *plot*(*C*(ω, 2), ω = 0 ..20, *color* = *black*, *title* = \`*practical resonance when c=2*\`);

practical resonance when c=2

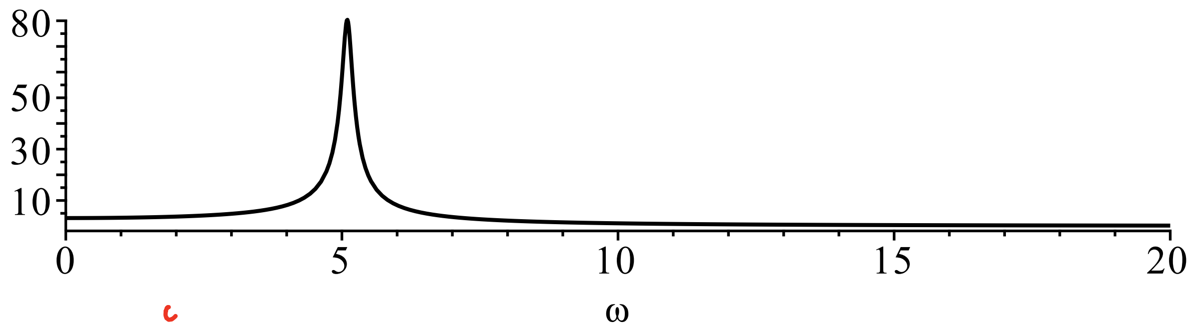


>

c =

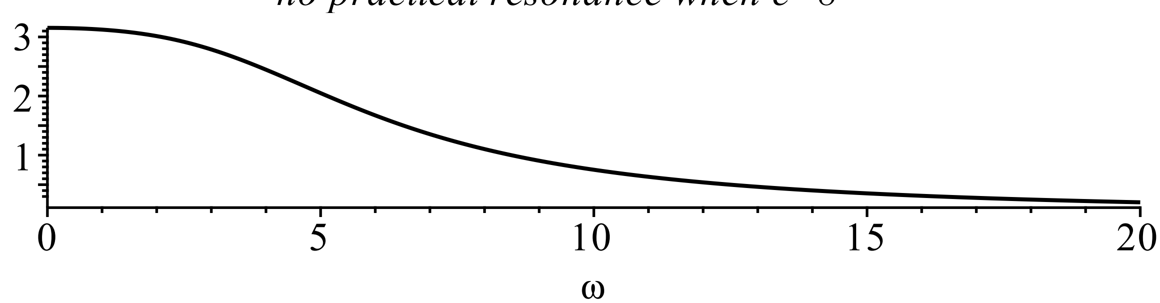
> *plot*(*C*(ω, .2), ω = 0 ..20, *color* = *black*, *title* = \`*practical resonance when c=0.2*\`);

practical resonance when c=0.2



> *plot*(*C*(ω, 8), ω = 0 ..20, *color* = *black*, *title* = \`*no practical resonance when c=8*\`);

no practical resonance when c=8



Math 2280-001
Week 8
Mon Feb 27

- Use last Friday's notes to understand damped forced oscillations,

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)$$

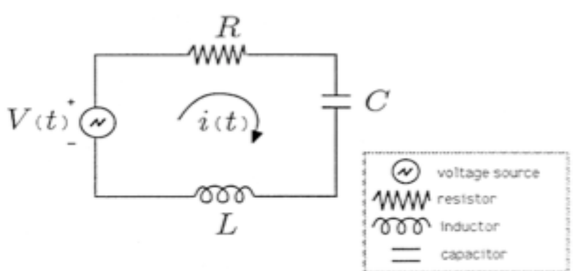
We will discuss undamped and damped forced oscillations.

$\boxed{\text{Wed Mar 1}}$

- Finish forced oscillations discussion if necessary.] Last week's notes

And then do something completely different, and entirely the same:
- 3.7 Electrical circuits.

It turns out that the mathematics of RLC circuits is identical to that for damped mass spring configurations:



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

Charge $Q(t)$ coulombs accumulates on the capacitor, at a rate (current) $I(t)$ ($i(t)$ in the diagram above) amperes (coulombs/sec), i.e $Q'(t) = I(t)$.

Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts). The units of voltage are energy units - Kirchoff's Law says that a test particle traversing any closed loop returns with the same potential energy level it started with:

For $Q(t)$: $\quad L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t)$ $\quad \frac{d}{dt}$ $\quad Q' = I$

For $I(t)$: $\quad L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t)$ $\quad m x'' + c x' + k x = f(t)$

If we specify that the voltage is given by a periodic *sine* function then $V'(t)$ will be a *cosine* function:

For $Q(t)$: $\quad L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t) = E_0 \sin(\omega\,t)$ $\quad V(t)$

For $I(t)$: $\quad L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t) = E_0\,\omega \cos(\omega\,t)$. $\quad \leftarrow V'(t)$

We can just transcribe all of our work from mass-spring systems. Just substitute L for m, R for c, $\frac{1}{C}$ for k, and $I(t)$ for $x(t)$. (There could be other situations where you want to study $Q(t)$ instead, depending on the context.)

$$L I''(t) + R I'(t) + \frac{1}{C} I(t) = V'(t) = E_0\,\omega \cos(\omega t)$$ $m x'' + c x' + k x = F_0 \cos \omega t$

For example, for *RLC* circuits with $R > 0$ the general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + I_{tr}(t)\,.$$

$$I_{sp}(t) = I_0 \cos(\omega t - \alpha) = I_0 \sin(\omega t - \gamma)\,, \quad \gamma = \alpha - \frac{\pi}{2}\,.$$

(One difference from the forced mass-spring configuration is that it makes sense to express $I_{sp}(t)$ as $I_0 \sin(\omega t - \gamma)$, since the applied voltage is $E_0 \sin(\omega t)$.) Since $0 \le \alpha \le \pi$, it follows that $-\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}$. Transcribing ...

$$\boxed{C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}} \Rightarrow \boxed{I_0(\omega) = \frac{E_0\,\omega}{\sqrt{\underbrace{\left(\frac{1}{C} - L\,\omega^2\right)^2}\, + R^2\,\omega^2}}}$$

$$\frac{E_0\,\omega}{=\omega\sqrt{\left(\frac{1}{C\omega} - L\omega\right)^2 + R^2}}$$

$$\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}}\,.$$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the <u>impedance</u> $Z(\omega)$ of the circuit (because the larger the impedance, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\boxed{\frac{1}{C\,\omega} = L\,\omega}$, i.e.

$$C = \frac{1}{L\,\omega^2}$$ if L is fixed and C is adjustable (old radios).

$$L = \frac{1}{C\,\omega^2}$$ if C is fixed and L is adjustable

Both L and C are adjusted in this M.I.T. lab demonstration:

http://www.youtube.com/watch?v=ZYgFuUl9_Vs.

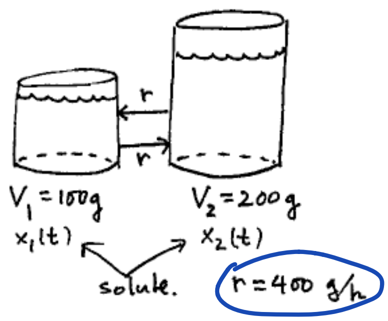
- Then begin the Friday notes, if we have time.

Math 2280-001
Fri Mar 3

4.1 Systems of differential equations - to model multi-component systems via compartmental analysis

http://en.wikipedia.org/wiki/Multi-compartment_model
and to understand higher order differential equations.

Here's a relatively simple 2-tank problem to illustrate the ideas:



Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$x_1'(t) = r_i c_i - r_o c_o$

$= 400 \cdot \frac{x_2}{200} - 400 \frac{x_1}{100} = 2x_2 - 4x_1 = -4x_1 + 2x_2$

$\left(\frac{g}{h}\right)\left(\frac{\text{mass}}{g}\right)$

$x_2'(t) = r_i c_i - r_o c_o$

$= 400 \frac{x_1}{100} - 400 \frac{x_2}{200} = 4x_1 - 2x_2$

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

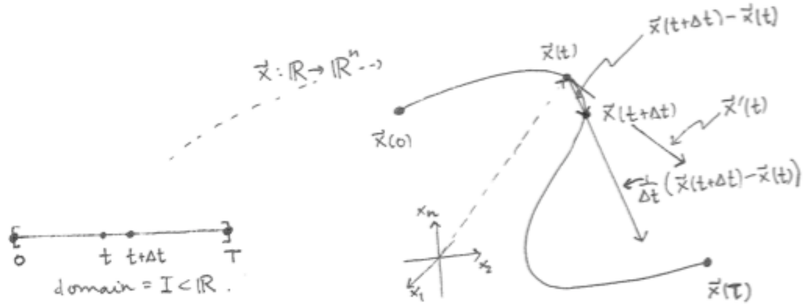
The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

$$\boxed{\begin{array}{c} \underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t)) \\ \underline{\boldsymbol{x}}\left(t_0\right) = \underline{\boldsymbol{x}}_0 \end{array}}$$

• We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

Why we expect IVP's for first order systems of DE's to have unique solutions $\underline{\boldsymbol{x}}(t)$:

• From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{\boldsymbol{x}}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{\boldsymbol{x}}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:



Analytically, the reason that the vector of derivatives $\underline{\boldsymbol{x}}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\underline{\boldsymbol{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ \vdots \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \right)$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\underline{\boldsymbol{x}}\left(t_0\right) = \underline{\boldsymbol{x}}_0$), and you know your "velocity" vector (depending on time and current location) $\Rightarrow$ you expect a unique solution! (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! And this is what numerical solvers do.)

These are vector analogs of the theorems we discussed in Chapter 1 for first order scalar differential equations. The first one should make intuitive sense, based on the reasoning of the previous page.

Theorem 1 For the IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}})$ is continuous in the $t-$variable and differentiable in its $\underline{\boldsymbol{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2 For the special case of the first order linear system of differential equations IVP

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. For example, with time-step h the Euler loop would increment as follows:

$$t_{j+1} = t_j + h$$
$$\underline{\boldsymbol{x}}_{j+1} = \underline{\boldsymbol{x}}_j + h\,\underline{\boldsymbol{F}}\left(t_j, \underline{\boldsymbol{x}}_j\right).$$

Remark: These theorems are the true explanation for why the n^{th}-order linear DE IVPs in Chapter 3 always have solutions - We will see that each $n^{th} - order$ linear DE IVP is actually equivalent to an IVP for a first order system of n linear DE's. (The converse is not true.) In fact, when software finds numerical approximations for solutions to higher order (linear or non-linear) DE IVPs that can't be found by the techniques of Chapter 3 or other mathematical formulas, it converts these IVPs to the equivalent first order system IVPs, and uses algorithms like Euler and Runge-Kutta to approximate the solutions.

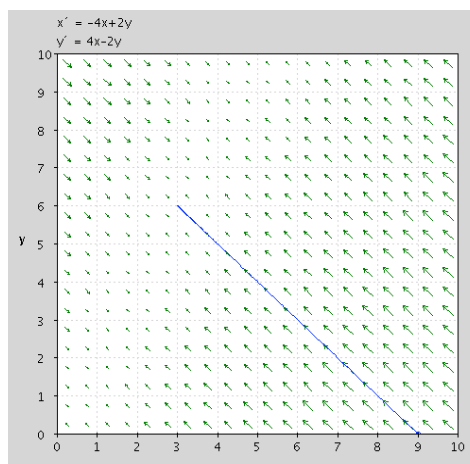
Exercise 2) Return to the page 1 tank example

$$x_1'(t) = -4\,x_1 + 2\,x_2$$
$$x_2'(t) = 4\,x_1 - 2\,x_2$$
$$x_1(0) = 9$$
$$x_2(0) = 0$$

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

2b) What are the apparent limiting solute amounts in each tank?

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?



<u>Exercise 7)</u> Consider an n^{th} order differential equation for a function $x(t)$:

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \ldots x^{(n-1)}(t)\right)$$

and the IVP

$$\begin{gathered} x^{(n)} = f\left(t, x, x', \ldots x^{(n-1)}\right) \\ x(t_0) = b_0 \\ x'(t_0) = b_1 \\ x''(t_0) = b_2 \\ \vdots \\ x^{(n-1)}(t_0) = b_{n-1}. \end{gathered}$$

<u>a)</u> Show that if $x(t)$ solves the IVP above, and if we define functions $x_1(t), x_2(t), \ldots\, x_n(t)$ by

$$x_1 := x,\ x_2 := x', x_3 := x'', \ldots\, x_n := x^{(n-1)}$$

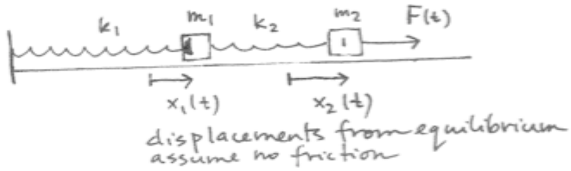
Then $\left[x_1, x_2, \ldots, x_n\right]$ solve the first order system IVP

$$\begin{gathered} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = f\left(t, x_1, x_2, \ldots x_{n-1}\right) \\ x_1(t_0) = b_0 \\ x_2(t_0) = b_1 \\ x_3(t_0) = b_2 \\ \vdots \\ x_n(t_0) = b_{n-1} \end{gathered}$$

<u>b)</u> Show that if $\left[x_1(t), x_2(t), \ldots x_n(t)\right]$ is a solution to the IVP in <u>a</u>, then the first function $x_1(t)$ solves the original IVP for the n^{th} order differential equtation.

Higher order systems of DE's are also equivalent to first order systems, as illustrated in the next example.

Consider this configuration of two coupled masses and springs:



Exercise 8) Use Newton's second law to derive a system of two second order differential equations for $x_1(t), x_2(t)$, the displacements of the respective masses from the equilibrium configuration. What initial value problem do you expect yields unique solutions in this case?

Exercise 9) Consider the IVP from Exercise 6, with the special values
$m_1 = 2,\ m_2 = 1;\ k_1 = 4,\ k_2 = 2;\ F(t) = 40\sin(3t)$:

$$\begin{aligned}
x_1'' &= -3x_1 + x_2 \\
x_2'' &= 2x_1 - 2x_2 + 40\sin(3t) \\
x_1(0) &= b_1,\ x_1'(0) = b_2 \\
x_2(0) &= c_1,\ x_2'(0) = c_2 .
\end{aligned}$$

9a) Show that if $x_1(t), x_2(t)$ solve the IVP above, and if we define

$$\begin{aligned}
v_1(t) &:= x_1'(t) \\
v_2(t) &:= x_2'(t)
\end{aligned}$$

then $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the first order system IVP

$$\begin{aligned}
x_1' &= v_1 \\
x_2' &= v_2 \\
v_1' &= -3x_1 + x_2 \\
v_2' &= 2x_1 - 2x_2 + 40\sin(3t) \\
x_1(0) &= b_1 \\
v_1(0) &= b_2 \\
x_2(0) &= c_1 \\
v_2(0) &= c_2 .
\end{aligned}$$

9b) Conversely, show that if $x_1(t), x_2(t), v_1(t), v_2(t)$ solve the IVP of four first order DE's, then $x_1(t), x_2(t)$ solve the original IVP for two second order DE's.