Math 2280-001 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we will cover. These notes are for sections 1.1-1.3, and part of 1.4.

Monday January ~~12~~ 9

- Go over course information on syllabus and course homepage:

http://www.math.utah.edu/~korevaar/2280spring17

- Notice that there is homework due this Friday, and our first quiz.

Then, let's begin!

Section 1.1 Introduction to differential equations

- What is an n^{th} order differential equation (DE)?

any equation involving a function $y = y(x)$ and its derivatives, for which the highest derivative appearing in the equation is the n^{th} one, $y^{(n)}(x)$; i.e. any equation which can be written as

$$F\left(x, y(x), y'(x), y''(x), ...y^{(n)}(x)\right) = 0.$$

Exercise 1: Which of the following are differential equations? For each DE determine the order.

a) For $y = y(x)$, $(y''(x))^2 + \sin(y(x)) = 0$ yes. 2^{nd} order

b) For $x = x(t)$, $x'(t) = 3\,x(t)\,(10 - x(t))$. yes. note, right-hand side need not be zero. 1^{st} order

c) For $x = x(t)$, $x' = 3\,x\,(10 - x)$. yes. abbreviated version of (b)

d) For $z = z(r)$, $z'''(r) + 4\,z(r)$. no. not an equation!

e) For $y = y(x)$, $y' = y^2$. yes. 1^{st} order

Definitions:

- A function $y(x)$ solves the differential equation $F\left(x, y, y', y'', y^{(n)}\right) = 0$ on some interval I (or is a solution function for the differential equation) means that $y(x)$ makes the differential equation a true equality for all x in I.

- A 1^{st} order DE is an equation involving a function and its first derivative. We may chose to write the function and variable as $y = y(x)$. In this case the differential equation is an equation equivalent to one of the form

$$F(x, y, y') = 0 .$$

Chapters 1-2 are about first order differential equations. For first order differential equations as above we can often use algebra to solve for y' in order to get what we call the **standard form** for the first order DE:

$$y' = f(x, y) .$$

- If we want our solution function to a first order DE to also satisfy $y\left(x_0\right) = y_0$, and if our DE is written in standard form, then we say that we are studying an **initial value problem** (IVP):

$$\text{IVP} \quad \begin{cases} y' = f(x, y) \\ y\left(x_0\right) = y_0 . \end{cases}$$

If we can find a solution function $y(x)$ that makes both equations of the the initial value problem true, then we say that $y(x)$ solves the initial value problem.

Exercise 2: Consider the differential equation $\dfrac{dy}{dx} = y^2$ from (1e).

a) Show that functions $y(x) = \dfrac{1}{C - x}$ solve the DE (on any interval not containing the constant C).

b) Find the appropriate value of C to solve the initial value problem

$$y' = y^2$$
$$y(1) = 2 .$$

a) If $y(x) = \dfrac{1}{C-x} = (C-x)^{-1}$

then $y'(x) = -(C-x)^{-2}(-1) = \dfrac{1}{(C-x)^2}$ (chain rule) } LHS of DE

compare to $y(x)^2 = \left(\dfrac{1}{C-x}\right)^2$ } RHS of DE

Since LHS = RHS the functions $y(x) = \dfrac{1}{C-x}$ make the DE a true equation so they are solutions

b) $y(x) = \dfrac{1}{C-x}$

$y(1) = \dfrac{1}{C-1} \overset{\text{want}}{=} 2$ so $C - 1 = \frac{1}{2} \Rightarrow C = 3/2$

$$\boxed{y(x) = \frac{1}{3/2 - x}}$$

2c) What is the largest interval on which your solution to 2b is defined as a differentiable function? Why?

interval must contain $x = 1$, so is $-\infty < x < 3/2$

2d) Do you expect that there are any other solutions to the IVP in 2b? Hint: The graph of the IVP solution function we found is superimposed onto a "slope field" below, where the line segment slopes at points (x, y) have values y^2 (because solution graphs to our differential equation will have those slopes, according to the differential equation). This might give you some intuition about whether you expect more than one solution to the IVP.

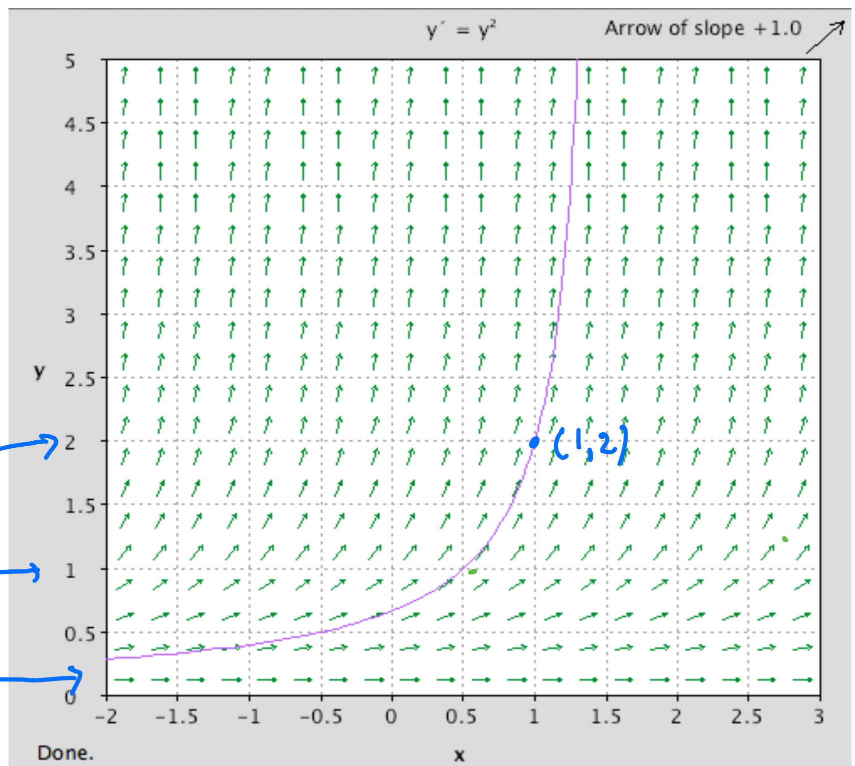
seems unlikely
since the graph $y = y(x)$
must contain $(1, 2)$, since $y(1) = 2$.
And then it must be tangent to the slope field

if for $y(x)$,
$y' = y^2$
then at any point (x, y)
on the graph of a
solution,
the slope of the
solution graph is given
by $y'(x)$, but also by y^2!

$y = 2$
$\Rightarrow$ slopes $y^2 = 4$

$y = 1$
$\Rightarrow$ slopes $y^2 = 1$

along x-axis
all slopes $= 0$



• **important course goals:** understand some of the key differential equations which arise in modeling real-world dynamical systems from science, mathematics, engineering; how to find the solutions to these differential equations if possible; how to understand properties of the solution functions (sometimes even without formulas for the solutions) in order to effectively model or to test models for dynamical systems.
In fact, you've encountered differential equations in previous mathematics and/or physics classes. For example:

• 1^{st} order differential equations: rate of change of function depends in some way on the function value, the variable value, and nothing else. For example, you've studied the population growth/decay differential equation for $P = P(t)$, and k a constant, given by

$P'(t)$ $=$

$$P'(t) = k\,P(t)$$

and having applications in biology, physics, finance. In this model, how fast the "population" changes is proportional to the population.

$= k \cdot P(t)$

note: two quantities are proportional means their ratio is a constant, i.e. one is a constant multiple of the other.

• 2^{nd} order DE's: Newton's second law (change in momentum equals net forces) often leads to second order differential equations for particle position functions $x = x(t)$ in physics.

$m\,x''(t) =$ net forces (could depend on x, x', t)

Exercise 3: The mathematical model in which the time rate of change of a population $P(t)$ is proportional to that population is expressed mathematically as

$$\frac{dP}{dt} = k\,P$$

where k is the proportionality constant.
3a) Find all solutions to this differential equation by using the chain rule backwards.
3b) The method of "separation of variables" is taught in most Calc I courses, and we'll cover it in detail in section 1.4. It's an algorithm which hides the "chain rule backwards" technique by treating the derivative $\frac{dP}{dt}$ as a quotient of differentials. Recall this magic algorithm to recover the solutions from 3a.

3a) $\frac{1}{P(t)} P'(t) = k$

antidiff wrt t:

$$\int \frac{P'(t)}{P(t)}\,dt = \int k\,dt$$

$u = P(t)$

$du = P'(t)\,dt$

$\int \frac{1}{u}\,du = \ln|u|$
$= \ln|P(t)|$

$\ln|P(t)| = kt + C$

exponentiate: $e^{\ln|P(t)|} = e^{kt+C} = e^{kt}e^{C}$

$|P(t)| = e^{C}e^{kt}$

$P(t) = C_1 e^{kt}$ $\quad (C_1 = \pm C \text{ or } -C)$

@ $t=0$, $P(0) = C_1 e^0 = C_1 \Rightarrow \boxed{P(t) = P(0)\,e^{kt}}$

write $\boxed{P(t) = P_0 e^{kt}}$

3b) differentials shortcut

$\frac{dP}{dt} = kP$

$\int \frac{dP}{P} = \int k\,dt$

$\ln|P| = kt + C$

Exercise 4) Newton's law of cooling is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ changes at a rate proportional to to the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use this model to derive the differential equation

$$\frac{dT}{dt} = -k(T - A) .$$

a) $T'(t) = \tilde{k}(T-A)$

$T > A$ expect $T' < 0$, cooling

$T' < 0 = \tilde{k}(T-A) = \tilde{k} \cdot$ positive

$\Rightarrow \tilde{k} < 0$

(also, $T < A$ expect $T' > 0$ $\Rightarrow \tilde{k} < 0$)

So rewrite as in (a), with $\tilde{k} = -k$, so now $k > 0$ (just because we like positive constants)

b) Would the model have been correct if we wrote $\frac{dT}{dt} = k(T - A)$ instead? yes

c) Use this model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70 ° F. An hour later the body temperature has decreased to 60 °. It's been a winter inversion in SLC, with constant ambient temperature 30 °. Assuming the Newton's law model, estimate the time of death.

ambient temp A

object $T(t)$

c) steps: solve DE:

$$\int \frac{1}{T-A}\, dT = \int -k\, dt$$

$$\ln|T-A| = -kt + C_1$$

$e \qquad e$

$$|T-A| = e^{-kt} e^{C_1}$$

$$T - A = Ce^{-kt} \qquad C = e^{C_1} \text{ or } -e^{C_1}$$

$$\boxed{T = A + Ce^{-kt}}$$

rest of steps: set $t = 0$ @ 3:00 pm. measure time in hours

$A = 30$

$T(t) = 30 + Ce^{-kt}$

$T(0) = 70$

$T(1) = 60$

use this info to find C & k, then set $T(t) = 98.6$ & solve for t.