Math 2280-001 Week 1 notes

We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we will cover. These notes are for sections 1.1-1.3, and part of 1.4.

Monday January ~~12~~ 9

- Go over course information on syllabus and course homepage:

http://www.math.utah.edu/~korevaar/2280spring17

- Notice that there is homework due this Friday, and our first quiz.

Then, let's begin!

Section 1.1 Introduction to differential equations

- What is an n^{th} order differential equation (DE)?

any equation involving a function $y = y(x)$ *and its derivatives, for which the highest derivative appearing in the equation is the* n^{th} *one,* $y^{(n)}(x)$ *; i.e. any equation which can be written as*

$$F\left(x, y(x), y'(x), y''(x), ... y^{(n)}(x)\right) = 0.$$

Exercise 1: Which of the following are differential equations? For each DE determine the order.

a) For $y = y(x)$, $\left(y''(x)\right)^2 + \sin(y(x)) = 0$ yes. 2nd order

b) For $x = x(t)$, $x'(t) = 3\,x(t)\,(10 - x(t))$.   yes. note, right-hand side need not be zero 1st order

c) For $x = x(t)$, $x' = 3\,x\,(10 - x)$.   yes. abbreviated version of (b)

d) For $z = z(r)$, $z'''(r) + 4\,z(r)$.   no. not an equation!

e) For $y = y(x)$, $y' = y^2$.   yes. 1st order

Definitions:

- A function $y(x)$ solves the differential equation $F(x, y, y', y'', y^{(n)}) = 0$ on some interval I (or is a solution function for the differential equation) means that $y(x)$ makes the differential equation a true equality for all x in I.

- A 1^{st} order DE is an equation involving a function and its first derivative. We may chose to write the function and variable as $y = y(x)$. In this case the differential equation is an equation equivalent to one of the form

$$F(x, y, y') = 0 .$$

Chapters 1-2 are about first order differential equations. For first order differential equations as above we can often use algebra to solve for y' in order to get what we call the **standard form** for the first order DE:

$$y' = f(x, y) .$$

- If we want our solution function to a first order DE to also satisfy $y(x_0) = y_0$, and if our DE is written in standard form, then we say that we are studying an **initial value problem** (IVP):

$$\text{IVP} \begin{cases} y' = f(x, y) \\ y(x_0) = y_0 . \end{cases}$$

If we can find a solution function $y(x)$ that makes both equations of the the initial value problem true, then we say that $y(x)$ solves the initial value problem.

Exercise 2: Consider the differential equation $\frac{dy}{dx} = y^2$ from (1e).

a) Show that functions $y(x) = \frac{1}{C - x}$ solve the DE (on any interval not containing the constant C).

b) Find the appropriate value of C to solve the initial value problem

$$y' = y^2$$
$$y(1) = 2 .$$

a) If $y(x) = \frac{1}{C-x} = (C-x)^{-1}$

then $\left.\begin{aligned} y'(x) &= -(C-x)^{-2}(-1) \quad \text{(chain rule)} \\ &= \frac{1}{(C-x)^2} \end{aligned}\right\}$ LHS of DE

compare to $\left. y(x)^2 = \left(\frac{1}{C-x}\right)^2 \right\}$ RHS of DE

Since LHS = RHS the functions $y(x) = \frac{1}{C-x}$ make the DE a true equation so they are solutions

b) $y(x) = \frac{1}{C-x}$

$y(1) = \frac{1}{C-1} \overset{\text{want}}{=} 2$ so $C - 1 = \frac{1}{2} \Rightarrow C = 3/2$

$$\boxed{y(x) = \frac{1}{3/2 - x}}$$

2c) What is the largest interval on which your solution to 2b is defined as a differentiable function? Why?

interval must contain $x = 1$, so is $-\infty < x < 3/2$

2d) Do you expect that there are any other solutions to the IVP in 2b? Hint: The graph of the IVP solution function we found is superimposed onto a "slope field" below, where the line segment slopes at points (x, y) have values y^2 (because solution graphs to our differential equation will have those slopes, according to the differential equation). This might give you some intuition about whether you expect more than one solution to the IVP.

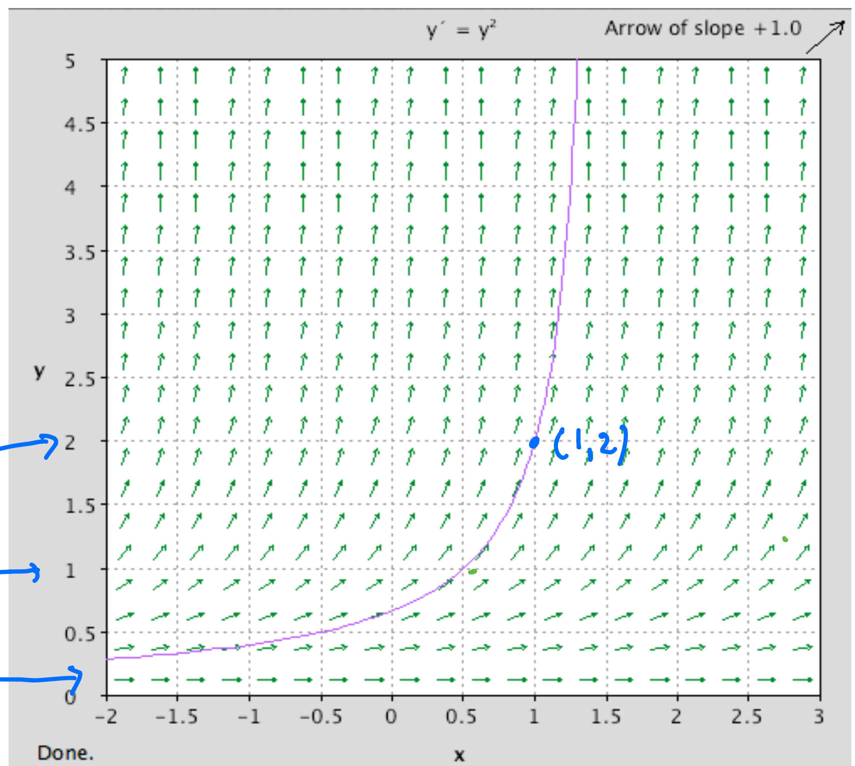
seems unlikely
since the graph $y = y(x)$
must contain $(1, 2)$, since $y(1) = 2$.
And then it must be tangent to the slope field

if for $y(x)$,
$y' = y^2$
then at any point (x, y)
on the graph of a
solution,
the slope of the
solution graph is given
by $y'(x)$, but also by y^2!

$y = 2$
$\Rightarrow$ slopes $y^2 = 4$

$y = 1$
$\Rightarrow$ slopes $y^2 = 1$

along x-axis
all slopes $= 0$



• **important course goals:** understand some of the key differential equations which arise in modeling real-world dynamical systems from science, mathematics, engineering; how to find the solutions to these differential equations if possible; how to understand properties of the solution functions (sometimes even without formulas for the solutions) in order to effectively model or to test models for dynamical systems.

In fact, you've encountered differential equations in previous mathematics and/or physics classes. For example:

• 1^{st} order differential equations: rate of change of function depends in some way on the function value, the variable value, and nothing else. For example, you've studied the population growth/decay differential equation for $P = P(t)$, and k a constant, given by

$$P'(t) = k\,P(t)$$

$P'(t)$ $=$

and having applications in biology, physics, finance. In this model, how fast the "population" changes is proportional to the population.

$= k \cdot P(t)$

note: two quantities are proportional means their ratio is a constant, i.e. one is a constant multiple of the other.

• 2^{nd} order DE's: Newton's second law (change in momentum equals net forces) often leads to second order differential equations for particle position functions $x = x(t)$ in physics.

$m\,x''(t) = \text{net forces}$ (could depend on x, x', t)

<u>Exercise 3</u>: The mathematical model in which the time rate of change of a population $P(t)$ is proportional to that population is expressed mathematically as

$$\frac{dP}{dt} = k\,P$$

where k is the proportionality constant.

<u>3a)</u> Find all solutions to this differential equation by using the chain rule backwards.

<u>3b)</u> The method of "separation of variables" is taught in most Calc I courses, and we'll cover it in detail in section 1.4. It's an algorithm which hides the "chain rule backwards" technique by treating the derivative $\frac{dP}{dt}$ as a quotient of differentials. Recall this magic algorithm to recover the solutions from <u>3a</u>.

3a) $\frac{1}{P(t)}P'(t) = k$

antidiff wrt t:

$$\int \frac{P'(t)}{P(t)}\,dt = \int k\,dt$$

$u = P(t)$

$du = P'(t)\,dt$

$\int \frac{1}{u}\,du = \ln|u| = \ln|P(t)|$

$$\ln|P(t)| = kt + C$$

exponentiate: $e^{\ln|P(t)|} = e^{kt+C} = e^{kt}e^{C}$

$|P(t)| = e^{C}e^{kt}$

$P(t) = C_1 e^{kt}$ $\quad(C_1 = \pm C \text{ or } -C)$

@ $t=0$, $P(0) = C_1 e^0 = C_1 \Rightarrow \boxed{P(t) = P(0)\,e^{kt}}$

write $\boxed{P(t) = P_0 e^{kt}}$

3b) differentials shortcut

$\frac{dP}{dt} = kP$

$\int \frac{dP}{P} = \int k\,dt$

$\ln|P| = kt + C$

Exercise 4) Newton's law of cooling is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ changes at a rate proportional ~~to~~ to the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use this model to derive the differential equation

$$\frac{dT}{dt} = -k(T - A)\,.$$

a) $T'(t) = \tilde{k}(T-A)$

$T > A$ expect $T' < 0$, cooling

$T' < 0 = \tilde{k}(T-A) = \tilde{k}\cdot\text{positive}$

$\Rightarrow \tilde{k} < 0$

(also, $T < A$ expect $T' > 0$ $\Rightarrow \tilde{k} < 0$)

So rewrite as in (a), with $\tilde{k} = -k$, so now $k > 0$ (just because we like positive constants)

b) Would the model have been correct if we wrote $\frac{dT}{dt} = k(T - A)$ instead? yes

c) Use this model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70° F. An hour later the body temperature has decreased to 60°. It's been a winter inversion in SLC, with constant ambient temperature 30°. Assuming the Newton's law model, estimate the time of death.

ambient temp A

object $T(t)$

c) steps: solve DE:

$$\int \frac{1}{T-A}\, dT = \int -k\, dt$$

$$\ln|T-A| = -kt + C_1$$

$$|T-A| = e^{-kt} e^{C_1}$$

$$T - A = Ce^{-kt} \qquad C = e^{C_1} \text{ or } -e^{C_1}$$

$$\boxed{T = A + Ce^{-kt}}\,.$$

rest of steps: set $t = 0$ @ 3:00 pm.
measure time in hours

$A = 30$

$T(t) = 30 + Ce^{-kt}$

$T(0) = 70$, $T(1) = 60$ } use this info to find C & k, then set $T(t) = 98.6$ & solve for t.

$$T(t) = 30 + 40e^{-kt}$$

$$T(1) = \cancel{30} + 40e^{-k\cdot 1} = \cancel{60}\ 30$$

$$e^{-k} = \frac{30}{40} = .75$$

$\ln[$

$$-k = \ln(.75) \Rightarrow k = -\ln(.75) = .2877$$

$$98.6 = 30 + 40e^{-kt}$$

solve for t: $t = -1.84$ hours.

@ 1.16 hours

1:096 o'clock.

Math 2280-1
Wed January 11
HW due Friday ...
Quiz Friday ...

It works!

- Review from Monday. What were the main ideas we talked about?

What is a DE.?
What is a soln to a DE?
What is an IVP (for a 1st order DE)
slope fields

- At the end of today's notes is the justification for "separation of variables". We will go over that at some point today.

Section 1.2: differential equations equivalent to ones of the form

$$y'(x) = f(x)$$

which we solve by direct antidifferentiation

$$y(x) = \int f(x)\,dx = F(x) + C.$$

Exercise 1 Solve the initial value problem

$$\frac{dy}{dx} = x\sqrt{x^2+4}$$

$$y(0) = 0$$

$$y = \int x\sqrt{x^2+4}\,dx = \frac{1}{3}(x^2+4)^{3/2} + C$$

$u = x^2+4$
$du = 2x\,dx$

$$\int u^{1/2}\,\tfrac{1}{2}\,du = \frac{1}{\cancel{2}}\,u^{3/2}\,\frac{\cancel{2}}{3} + C$$

shortcut:
ans is mult of $(x^2+4)^{3/2}$

$D_x \frac{1}{3}(x^2+4)^{3/2} = \frac{1}{3}\frac{3}{\cancel{2}}(x^2+4)^{1/2}\cancel{2}x$
$\cancel{\frac{1}{3}}\cancel{3}\,x(x^2+4)^{1/2}$

$$y(0) = 0 = \frac{4^{3/2}}{3} + C$$

$$0 = \frac{8}{3} + C \Rightarrow C = -8/3$$

$$\boxed{y(x) = \frac{1}{3}(x^2+4)^{3/2} - 8/3}$$

An important class of such problems arises in physics, usually as velocity/acceleration problems via Newton's second law. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t) .$$

$x(t)$ = position
$x'(t) = v(t)$ vel.
$v'(t) = a(t)$ accel.

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

$$m\,x''(t) = f(t) \quad \text{net force}$$
$$x''(t) = \tfrac{1}{m} f(t)$$

Exercise 2:
a) If the units for position are meters m and the units for time are seconds s, what are the units for velocity and acceleration? (These are *mks* units.)
b) Same question, if we use the English system in which length is measured in feet and time in seconds. Could you convert between *mks* units and English units?

$x(t)$ length

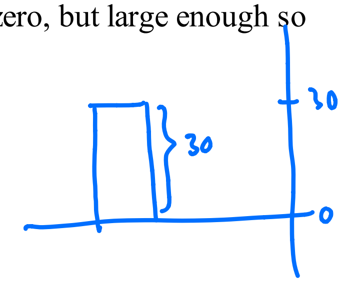
$$v(t) = x'(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \qquad \frac{\text{length}}{\text{time}}$$

$$a(t) = v'(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} \qquad \frac{\text{length/time}}{\text{time}} = \frac{\text{length}}{\text{time}^2}$$

	mks	English
x	m	ft
v	m/s	ft/s
a	m/s²	ft/s²

Exercise 3: A projectile with very low air resistance is fired almost straight up from the roof of a building 30 meters high, with initial velocity 50 m/s. Its initial horizontal velocity is near zero, but large enough so that the object lands on the ground rather than the roof.

a) Neglecting friction, how high will the object get above ground?
b) When does the object land ?



$g = 9.8 \ m/s^2$

$$m y'' = -mg$$
$$\int y'' = \int -g$$
$$v(t) = y'(t) = \int -g\,dt = -gt + C .$$
@ $t=0$: $y'(0) = 0 + C$ so we call this v_0
$$\int y'(t) = \int -gt + v_0 \, dt$$
$$y(t) = -g\frac{t^2}{2} + v_0 t + C$$
@ $t=0$: $y(0) = 0 + 0 + C$ call this const y_0

$$\boxed{y(t) = -\frac{g}{2}t^2 + v_0 t + y_0}$$

for us: $\boxed{y(t) = -4.9t^2 + 50t + 30}$

$\boxed{v(t) = -9.8t + 50}$

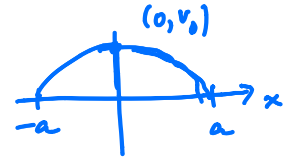
a) at y_{max}, $v(t) = 0$
$-9.8t + 50 = 0$
$t = \frac{50}{9.8} = 5.097$
$y(t) = 157.42$ m.

b) Solve $y(t) = 0$ for t
10.77 sec.

Here's another fun example from section 1.2, which also reviews important ideas from Calculus - in particular we will see how the fact that the slope of a graph $y = g(x)$ is the derivative $\frac{dy}{dx}$ can lead to first order differential equations.

<u>Exercise 4</u>: (See "A swimmer's Problem" and Example 4 in section 1.2). A swimmer wishes to cross a river of width $w = 2\,a$, by swimming directly towards the opposite side, with constant transverse velocity v_S. The river velocity is fastest in the middle and is given by an even function of x, for $-a \le x \le a$. The velocity equal to zero at the river banks. For example, it could be that

river velocity $$v_R(x) = v_0\left(1 - \frac{x^2}{a^2}\right).$$



See the configuration sketches below.

a) Writing the swimmer location at time t as $(x(t), y(t))$, translate the information above into expressions for $x'(t)$ and $y'(t)$.

b) The parametric curve describing the swimmer's location can also be expressed as the graph of a function $y = y(x)$. Show that $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right).$$

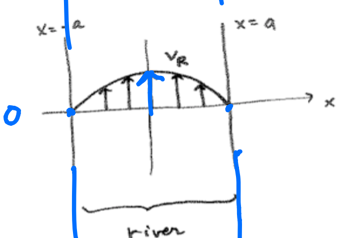
c) Compute an integral or solve a DE, to figure out how far downstream the swimmer will be when she reaches the far side of the river.

$\boxed{\frac{4}{3}a\frac{v_0}{v_S}}$ makes sense

swimmer vel.

$x'(t) = v_S$
$y'(t) = v_R(x(t))$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$



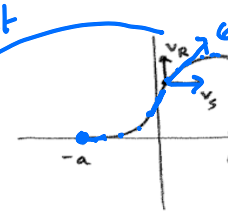


image of parametric curve $(x(t), y(t))$ is also graph $y = y(x)$

$$\frac{d}{dt}\, y(x(t)) = \frac{dy}{dx}\,\frac{dx}{dt}$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}\cdot\frac{\cancel{dx}}{\cancel{dt}}$$

(b) $$\boxed{\frac{dy}{dx} = \frac{v_R}{v_S} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)}$$ §1.2

(c): antidiff to find $y(x)$. With initial condition $y(-a) = 0$. Then compute $y(a)$.

(c'): shortcut: go $y(a) - y(-a)$ downriver
$$= \int_{-a}^{a} y'(x)\,dx$$

$$= \int_{-a}^{a} \frac{v_0}{v_s} \underbrace{\left(1 - \frac{x^2}{a^2}\right)}_{\text{even}} dx = 2\frac{v_0}{v_s}\left[x - \frac{x^3}{3a^2}\right]_{0}^{a} =$$

$$= 2\int_0^a \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right) dx = 2\frac{v_0}{v_s}\left[a - \frac{a^3}{3a^2}\right]$$

$$= 2\frac{v_0}{v_s}\frac{2}{3}a$$

$$\boxed{= \frac{4}{3} a \frac{v_0}{v_s}}$$

Exercise 5:

Suppose the acceleration function is a negative constant $-a$,

$$x''(t) = -a .$$

(This could happen for vertical motion, e.g. near the earth's surface with $a = g \approx 9.8 \frac{m}{s^2} \approx 32 \frac{ft}{s^2}$, as well as in other situations.)

a) Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.

b) Assuming $x(0) = 0$ and $v_0 > 0$, show that the maximum value of $x(t)$ is

$$x_{max} = \frac{1}{2}\frac{v_0^2}{a} .$$

(This formula may help with some homework problems.)

(skip)

1.4 Separable DE's: Important applications, as well as a lot of the examples we study in slope field discussions of section 1.3 are separable DE's. So let's discuss precisely what they are, and why the separation of variables algorithm works.

<u>Definition:</u> A separable first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\boxed{\frac{dy}{dx} = f(x)\phi(y)\,.} \leftarrow$$

It's more convenient to rewrite this DE as

$$\frac{1}{\phi(y)}\frac{dy}{dx} = f(x), \qquad \text{(as long as } \phi(y) \neq 0\text{)}\,.$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$\boxed{g(y)\frac{dy}{dx} = f(x)\,.} \leftarrow \text{integrate both sides wrt } x$$

<u>Solution (math justified)</u>: The left side of the modified differential equation is short for $g(y(x))\dfrac{dy}{dx}$. And if $G(y)$ is any antiderivative of $g(y)$, then we can rewrite this as

$$G'(y(x))y'(x)$$

which by the chain rule (read backwards) is nothing more than

$$\frac{d}{dx}G(y(x))\,.$$

And the solutions to

$$\frac{d}{dx}G(y(x)) = f(x)$$

are

$$\boxed{G(y(x)) = \int f(x)\,dx = F(x) + C.}$$

$$\int g(y(x))\frac{dy}{dx}\,dx = \int f(x)\,dx$$

$u = y(x)$

$du = y'(x)dx$

$\int g(u)\,du$

$G(u) + C_1$

$G(y(x)) + C_1 = F(x) + C_2$

where $F(x)$ is any antiderivative of $f(x)$. Thus solutions $y(x)$ to the original differential equation satisify

$$G(y) = F(x) + C\,. \leftarrow \text{solns } y(x) \text{ solve this implicit eqtn}$$

This expresses solutions $y(x)$ <u>implicitly</u> as functions of x. You may be able to use algebra to solve this equation <u>explicitly</u> for $y = y(x)$, and (working the computation backwards) $y(x)$ will be a solution to the DE. (Even if you can't algebraically solve for $y(x)$, this still yields implicitly defined solutions.)

<u>Solution (differential magic)</u>: Treat $\dfrac{dy}{dx}$ as a quotient of differentials dy, dx, and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}\,.$$

$$\int g(y)dy = \int f(x)dx\,.$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx\,, \text{ i.e.}$$

$$G(y) + C_1 = F(x) + C_2 \Rightarrow \boxed{G(y) = F(x) + C\,.} \quad \text{Agrees!}$$

This is the same differential magic that you used for the "method of substitution" in antidifferentiation, which was essentially the "chain rule in reverse" for integration techniques.

<u>Ex</u> $\quad \dfrac{dy}{dx} = \dfrac{x}{y^2} \qquad \int y^2\,dy = \int x\,dx$

$\dfrac{1}{3}y^3 = \dfrac{1}{2}x^2 + C$, then if want to, solve explicitly for y

Math 2280-001
Fri Jan 13
1.3-1.4 more slope fields; existence and uniqueness for solutions to IVPs; using separable differential equations for examples.

Next week, bring your own notes. Posted by 3:00 today

Slope fields
$\exists!$ theorem §1.3

- Quiz at the end of class on sections 1.1-1.2

If $y(x)$ is a solution to this IVP and if we consider its graph $y = y(x)$, then the IC means the graph must pass through the point (x_0, y_0). The DE means that at every point (x, y) on the graph the slope of the graph must be $f(x, y)$. (So we often call $f(x, y)$ the "slope function" for the differential equation.) This gives a way of understanding the graph of the solution $y(x)$ even without ever actually finding a formula for $y(x)$! Consider a **slope field** near the point (x_0, y_0): at each nearby point (x, y), assign the slope given by $f(x, y)$. You can represent a slope field in a picture by using small line segments placed at representative points (x, y), with the line segments having slopes $f(x, y)$.

Exercise 1: Consider the differential equation $\frac{dy}{dx} = x - 3$, and then the IVP with $y(1) = 2$.

a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \le x \le 5,\ 0 \le y \le 6$. Notice that in this example the value of the slope field only depends on x, so that all the slopes will be the same on any vertical line (having the same x-coordinate). (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started, see below.
b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without resorting to a formula for the solution function $y(x)$.
c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).

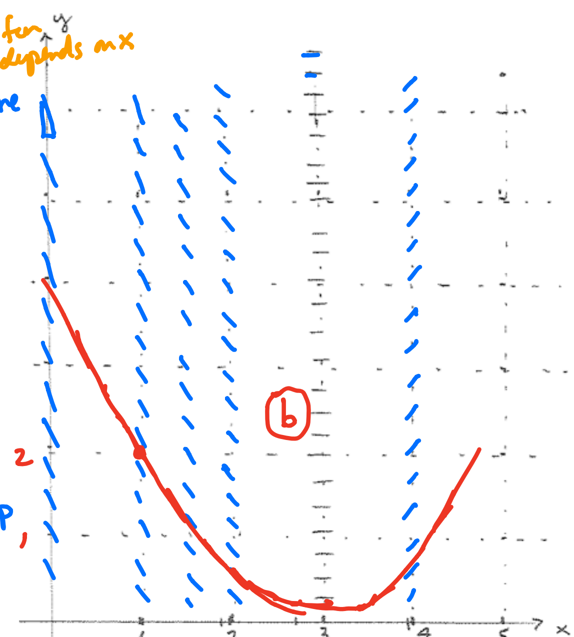
$\frac{dy}{dx} = f(x,y)$
soln $y = y(x)$ makes DE true.
2 ways to compute $y'(x) \to$ slope of graph $y = y(x)$ @ (x,y)
1) take deriv of $y(x)$
2) evaluate $f(x,y)$, $y = y(x)$

Slope fields: sketch field of slopes $f(x,y)$
Every graph of a soln has to be tangent to slope field.

$$\begin{cases} y'(x) = x - 3 \\ y(1) = 2 \end{cases}$$

slope fcn only depends on x

x	slope
3	0
2	-1
4	1
1	-2
0	-3
1.5	-1.5



(c) Solve DE/IVP analytically.

$y'(x) = x - 3$
$\Rightarrow y(x) = \frac{x^2}{2} - 3x + C$
$y(1) = 2 \Rightarrow 2 = \frac{1}{2} - 3 + C$
$\frac{9}{2} = C$

$y(x) = \frac{x^2}{2} - 3x + \frac{9}{2}$
$= \frac{1}{2}(x^2 - 6x + 9)$
$y = \frac{1}{2}(x-3)^2$

graph is parabola with vertex at $(3,0)$

The procedure of drawing the slope field $f(x, y)$ associated to the differential equation $y'(x) = f(x, y)$ can be automated. And, by treating the slope field as essentially constant on small scales, i.e. using

$$\frac{\triangle y}{\triangle x} \approx \frac{dy}{dx} = f(x, y)$$

one can make discrete steps in x and y, starting from the initial point (x_0, y_0), by picking a step size $\triangle x$ and then incrementing y by

$$\triangle y = f(x, y)\underbrace{\triangle x}_{\text{fix}}.$$

In this way one can *approximate* solution functions to initial value problems, and their graphs. The Java applet "dfield" (stands for "direction field", which is a synonym for slope field) uses (a more sophisticated analog of) this method to compute approximate solution graphs.

Here's a picture like the one we sketched by hand on the previous page, created by dfield.

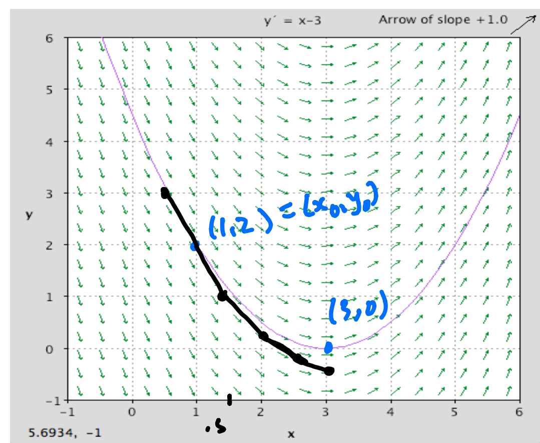
(x_0, y_0)

$x_1 = x_0 + \Delta x \qquad y_1 = x_0 + \Delta y$

$x_2 = x_0 + 2\Delta x \qquad \quad = x_0 + f(x_0, y_0)\Delta x$

$\vdots$

we come back to this in §2.4-2.6



Exercise 2: Consider the IVP

$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

a) Check that $y(x) = x + 1 + C\,e^x$ gives a family of solutions to the DE (C=const). Notice that we haven't yet discussed a method to derive these solutions, but we can certainly check whether they work or not.

b) Solve the IVP by choosing appropriate C.

c) Sketch the solution by hand, for the rectangle $-3 \le x \le 3, -3 \le y \le 3$. Also sketch typical solutions for several different C-values. Notice that this gives you an idea of what the slope field looks like. How would you attempt to sketch the slope field by hand, if you didn't know the general solutions to the DE? What are the isoclines in this case?

d) Compare your work in (c) with the picture created by dfield on the next page.

a): plug $y(x)$ into DE, see if get identity.

$$\boxed{y(x) = x + 1 + Ce^x}$$

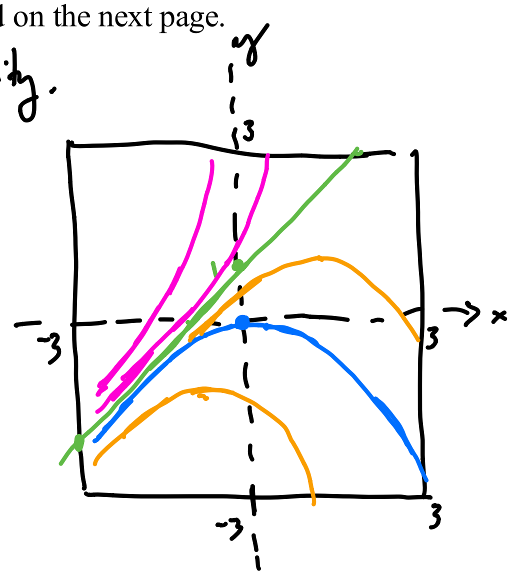
$C=0$
$C<0$
$C>0$

LHS $y' = 1 + Ce^x$

RHS $y - x = (\cancel{x} + 1 + Ce^x) - \cancel{x}$

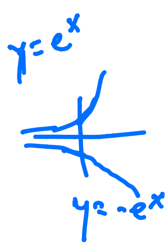
LHS=RHS $\Rightarrow$ DE true.

c)



b) $y(0) = 0 \Rightarrow \quad 0 = 1 + C$
$\quad -1 = C$

$$\boxed{y(x) = x + 1 - e^x}$$



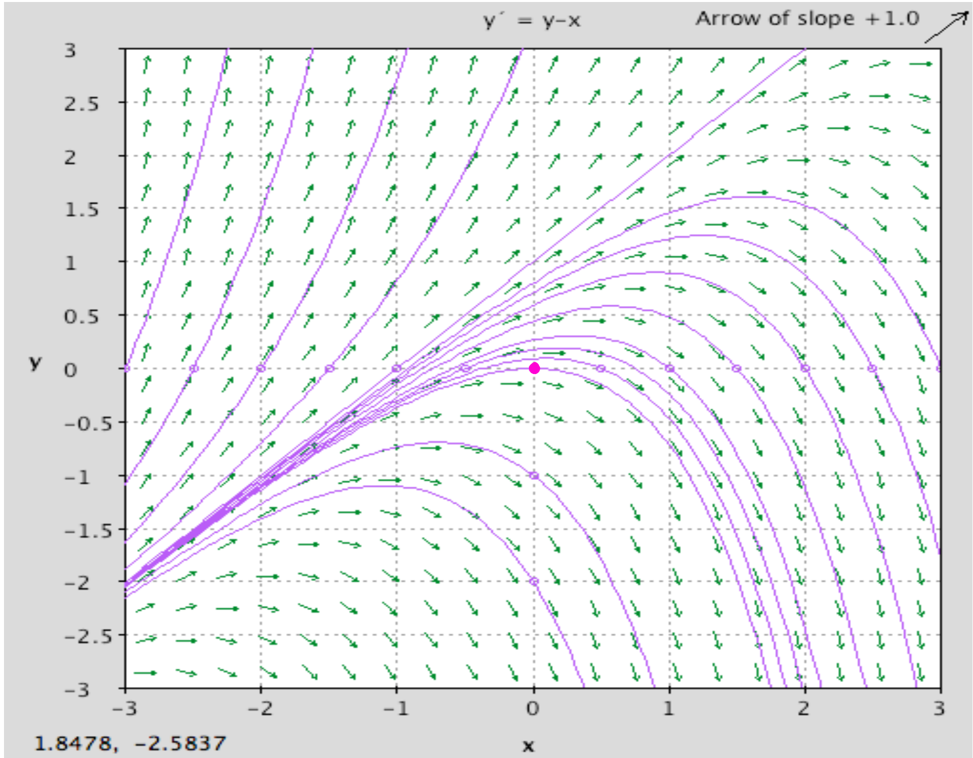
as $x \to -\infty$

$y(x) \approx x + 1$

$y = x + 1$

diag asymp

for all soltns, as $x \to -\infty$



y´ = y−x
Arrow of slope +1.0
y
x
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−3
−2
−1
0
1
2
3
1.8478, −2.5837

Exercise 3: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 .$$

a) Use separation of variables to find solutions to this DE. } you do

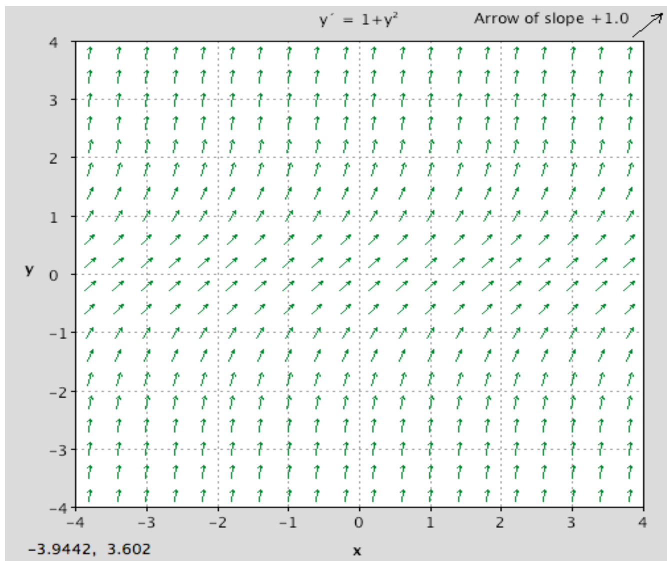
b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a? (You can sketch by hand, I'll use "dfield" on my browser.)

c) Explain why each IVP has a solution, but this solution does not exist for all x.

You can download the java applet "dfield" from the URL

http://math.rice.edu/~dfield/dfpp.html

(You also have to download a toolkit, following the directions there.)



Exercise 4a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$\boxed{y(0) = 0}$$

4b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called <u>singular</u> solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

4c) Sketch some of these singular solutions onto the slope field below.

4a) $\int \frac{dy}{y^{2/3}} = \int dx \leftarrow$ assumed $y^{2/3} \neq 0$ Case 1

$\int y^{-2/3}\,dy = \int dx$

$\boxed{3y^{1/3} = x + C}$

$\boxed{y^{1/3} = \frac{1}{3}x + C} \leftarrow$

IVP $y(0) = 0 \Rightarrow 0 = 0 + C$

$\Rightarrow C = 0$

$y^{1/3} = \frac{1}{3}x$

$y = \frac{x^3}{27}$

x	y
0	0
3	1
-3	-1

Case 1 get solns

$y = \left[\frac{1}{3}(x+C)\right]^3$

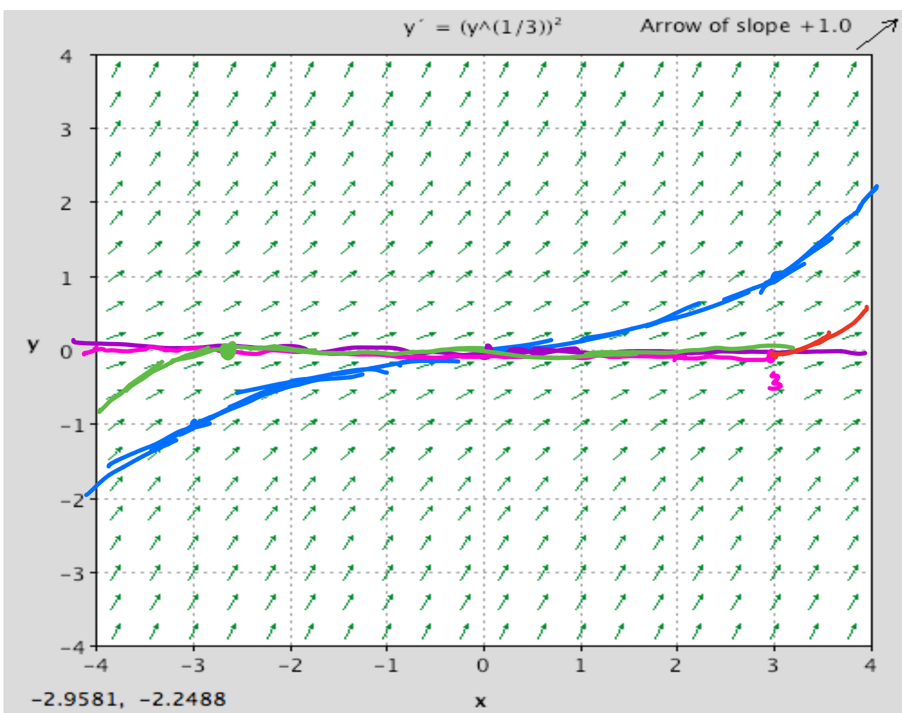
$y = \frac{1}{27}(x+C)^3$

Case 2 : $y = 0$

if $y(x) \equiv 0$, that's a soln

check $\Rightarrow y' = 0$ LHS

$(y(x))^{2/3} = 0$ RHS



3rd soln:

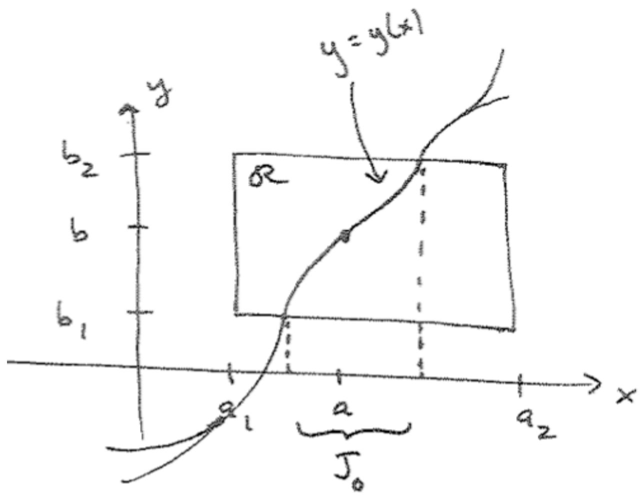
$$y(x) = \begin{cases} 0 & x \leq 3 \\ \frac{1}{27}(x-3)^3 & x \geq 0 \end{cases}$$

Here's what's going on (stated in 1.3 page 24 of text; partly proven in Appendix A.)
<u>Existence - uniqueness theorem for the initial value problem</u>
Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

- Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \le x \le a_2, b_1 \le y \le b_2$ in the $x-y$ plane.
- <u>Existence:</u> If $f(x, y)$ is <u>continuous</u> in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.
- <u>Uniqueness:</u> If the partial derivative function $\boxed{\frac{\partial}{\partial y} f(x, y) \text{ is also continuous in } \mathcal{R},}$ then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



<u>Exercise 5</u>: Discuss how the existence-uniqueness theorem is consistent with our work in Exercises 1-4 in today's notes, where we were able to find explicit solution formulas because the differential equations were actually separable (#1,3,4) or when the solution formula was given to us (#2).