Start here Monday

2.3 Improved velocity models: velocity-dependent drag forces

For particle motion along a line, with

$$\text{position } x(t) \text{ (or } y(t)\text{)},$$
$$\text{velocity } x'(t) = v(t), \text{ and}$$
$$\text{acceleration } x''(t) = v'(t) = a(t)$$

We have Newton's 2^{nd} law

$$m\,v'(t) = F \quad = \text{e.g. } -mg \quad = \text{e.g. pure drag force prop. to velocity}$$

where F is the net force.

- We're very familiar with constant force $F = m\,\alpha$, where α is a constant:

$$v'(t) = \alpha$$
$$v(t) = \alpha\,t + v_0$$
$$x(t) = \frac{1}{2}\alpha\,t^2 + v_0\,t + x_0 .$$

Examples we've seen a lot of:

- $\alpha = -g$ near the surface of the earth, if up is the positive direction, or $\alpha = g$ if down is the positive direction.
- boats or cars or "particles" subject to constant acceleration or deceleration.

New today !!! Combine a constant force with a velocity-dependent drag force, at the same time. The text calls this a "resistance" force:

$$m\,v'(t) = m\,\alpha + F_R$$

Empirically/mathematically the resistance forces F_R depend on velocity, in such a way that their magnitude is

$$|F_R| \approx k\,|v|^p,\ 1 \le p \le 2 . \qquad p = 1 \quad p = 2$$

- $p = 1$ (linear model, drag proportional to velocity):

$$m\,v'(t) = m\,\alpha - k\,v$$

This linear model makes sense for "slow" velocities, as a linearization of the frictional force function, assuming that the force function is differentiable with respect to velocity...recall Taylor series for how the velocity resistance force might depend on velocity:

$$F_R(v) = \cancel{F_R(0)}^{\,v=0} + F_R'(0)\,v + \frac{1}{2!}F_R''(0)\,v^2 + \dots \frac{1}{3!}F_R'''(0)\,v^3 + \dots$$

$\boxed{F_R(0) = 0}$ and for small enough v the higher order terms might be negligable compared to the linear term, so "linearization"

$$F_R(v) \approx F_R'(0)\,v \approx -k\,v .$$

We write $-k\,v$ with $k > 0$, since the frictional force opposes the direction of motion, so sign opposite of the velocity's.

http://en.wikipedia.org/wiki/Drag_(physics)#Very_low_Reynolds_numbers:_Stokes.27_drag

$\cancel{m}v' = \frac{\tilde{\alpha}}{m} - \frac{k}{m}v$

Exercise 1a: Rewrite the linear drag model as

$$v'(t) = \alpha - \rho\, v$$

equil soltn: set $\alpha - \rho v = 0$ *

$-\rho\left(v - \frac{\alpha}{\rho}\right) = 0$ *

where the $\boxed{\rho = \frac{k}{m}}$. Construct the phase diagram for v. Notice that $v(t)$ has exactly one constant (equilibrium) solution, and find it. Its value is called the *terminal velocity*. Explain why *terminal velocity* is an appropriate term of art, based on your phase diagram.

$v = \frac{\alpha}{\rho}$

$v < \alpha/\rho$
$v' = (-)(-) = +$
$\Rightarrow v$ inc.

$v > \alpha/\rho$
$v' = (-)(+) = - \Rightarrow v$ is dec.

α/ρ asymptotically stable

from last week, $\lim_{t\to\infty} v(t) = \alpha/\rho$

no matter what v_0 is!

call $\frac{\alpha}{\rho} = v_T$ = "terminal velocity"

1b) Solve the IVP

$$v'(t) = \alpha - \rho\, v$$
$$v(0) = v_0$$

and verify your phase diagram analysis. (This is, once again, our friend the first order constant coefficient linear differential equation.)

$v' + \rho v = \alpha$

$P(t) = \rho$

I.F. $e^{\int P(t)dt} = e^{\rho t}$

$e^{\rho t}\left[v' + \rho v\right] = \alpha e^{\rho t}$

$\frac{d}{dt}\left(e^{\rho t} v\right) = \alpha e^{\rho t}$

$e^{\rho t} v = \int \alpha e^{\rho t}\, dt = \frac{\alpha}{\rho} e^{\rho t} + C$

$\div e^{\rho t}$ $\boxed{v = \frac{\alpha}{\rho} + Ce^{-\rho t}}$

@ $t=0$, $v = v_0 = \frac{\alpha}{\rho} + C \Rightarrow C = v_0 - \alpha/\rho$

1c) integrate the velocity function above to ~~find a formula for the~~ position function $y(t)$.

$\boxed{v(t) = \frac{\alpha}{\rho} + \left(v_0 - \frac{\alpha}{\rho}\right)e^{-\rho t}}$

$\boxed{v(t) = v_T + (v_0 - v_T)e^{-\rho t}}$

$y(0) = y_0$.

$y(t) = \int v(t)\, dt$

$= v_T t + \frac{(v_0 - v_T)}{-\rho} e^{-\rho t} + \boxed{C}$

@ $t = 0$: $y_0 = 0 + \frac{v_0 - v_T}{-\rho} \cdot 1 + C \Rightarrow \boxed{C = y_0 + \frac{v_0 - v_T}{\rho}}$

$$\boxed{y(t) = y_0 + v_T t + \frac{v_0 - v_T}{\rho}\left[1 - e^{-\rho t}\right]}$$

$\boxed{v_T = \frac{\alpha}{\rho}}$

initial value

const. term. velocity term

what's left (if $v_0 = v_T$, term disappears)

- $p = 2$, for the power in the resistance force. This can be an appropriate model for velocities which are not "near" zero....described in terms of "Reynolds number". Accounting for the fact that the resistance opposes direction of motion we get

$$\cancel{m}\, v'(t) = \cancel{m}\, \alpha - \frac{k}{m} v^2 \quad \text{if } v > 0\,, \quad F_r < 0$$
$$\cancel{m}\, v'(t) = \cancel{m}\, \alpha + \frac{k}{m} v^2 \quad \text{if } v < 0. \quad F_r > 0$$

http://en.wikipedia.org/wiki/Drag_(physics)#Drag_at_high_velocity

Exercise 2) Once again letting $\boxed{\rho = \frac{k}{m}}$ we can rewrite the DE's as

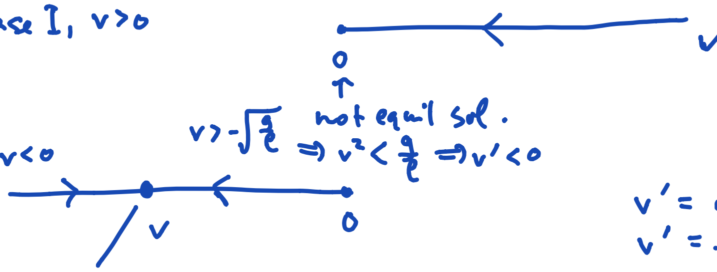
$$v'(t) = \alpha - \rho\, v^2 \quad \text{if } v > 0$$
$$v'(t) = \alpha + \rho\, v^2 \quad \text{if } v < 0.$$

$v' = -g - \rho v^2$

2a) Consider the case in which $\boxed{\alpha = -g}$, so we are considering vertical motion, with up being the positive direction. Draw the phase diagrams. Note that each diagram contains a half line of v-values. Make conclusions about velocity behavior in case $v_0 > 0$ and $v_0 \le 0$. Is there a terminal velocity?

$v' < 0 \Rightarrow v$ dec.

case I, $v > 0$



$v > -\sqrt{\frac{g}{\rho}} \Rightarrow v^2 < \frac{g}{\rho} \Rightarrow v' < 0$

case II, $v < 0$

$v < -\sqrt{\frac{g}{\rho}}$
$v^2 > \frac{g}{\rho}$
$v' > 0$

$-\sqrt{\frac{g}{\rho}}$ asymptotically stable.

$\boxed{v_\tau := -\sqrt{\frac{g}{\rho}}}$

$v' = \alpha + \rho v^2$
$v' = -g + \rho v^2$
$v' = \rho\left(v^2 - \frac{g}{\rho}\right)$
roots: $v = \pm\sqrt{\frac{g}{\rho}}$
$\left(v^2 = g/\rho\right)$

2b) Set up the two separable differential equation IVPs for the cases above, so that you will be able to complete finding the solution formulas (in your homework)....Of course, once you find the velocity function you'll still need to integrate that, if you want to find the position function!

$\lim_{t \to \infty} v(t) = v_\tau\,,$ for all v_0.

$v > 0$: $v'(t) = -g - \rho v^2$

$$\frac{dv}{dt} = -\rho\left(v^2 + \frac{g}{\rho}\right)$$

$$\int \frac{dv}{v^2 + \frac{g}{\rho}} = \int -\rho\, dt$$

u-substitution then arctan....

$v < 0$: $v' = -g + \rho v^2$

$$\frac{dv}{dt} = \rho\left(v^2 - \frac{g}{\rho}\right)$$

$$\frac{dv}{v^2 - g/\rho} = \rho\, dt$$

$$\frac{dv}{\left(v - \sqrt{g/\rho}\right)\left(v + \sqrt{\frac{g}{\rho}}\right)} = \rho\, dt$$

partial fractions

Application: We consider the bow and deadbolt example from the text, page 102-104. It's shot vertically into the air (watch out below!), with an initial $\underbrace{\text{velocity of } 49\ \frac{m}{s}}_{}$ "5g". In the no-drag case, this could just be the vertical component of a deadbolt shot at an angle. With drag, one would need to study a more complicated system of DE's for the horizontal and vertical motions, if you didn't shoot the bolt straight up.

<u>Exercise 3</u>: First consider the case of no drag, so the governing equations are

$$v'(t) = -g \approx -9.8\ \frac{m}{s^2}$$

$$v(t) = -g\,t + v_0 = -g\,t + 5\,g \quad g\,[-t+5]$$

$$x(t) = -\frac{1}{2}g\,t^2 + v_0\,t + \overset{=0}{x_0} = -\frac{1}{2}g\,t^2 + 5\,g\,t. = -\frac{1}{2}g\,(t^2-10t) = -\frac{1}{2}g\,t\,(t-10)$$

Find when $v = 0$ and deduce how long the object rises, how long it falls, and its maximum height.

set $v(t)=0$ solve for t $t=5$ sec.

set $x(t)=0$ solve for t $t=10$ sec.

$x(5) = -\frac{1}{2}g\cdot 5\cdot(-5)$
$= \frac{25}{2}g$
$= \frac{25}{2}\cdot 9.8$
≈ 120
122.5

Maple check:

```
> restart :
  Digits := 5 :
```

```
> g := 9.8;
  v0 := 49.0;
  v1 := t -> -g·t + v0;
  y1 := t -> -1/2·g·t^2 + v0·t;
```

$$g := 9.8$$

$$v0 := 49.0$$

$$v1 := t \rightarrow -g\,t + v0$$

$$y1 := t \rightarrow -\frac{1}{2}\,g\,t^2 + v0\,t \qquad (6)$$

Exercise 4: Now consider the linear drag model for the same deadbolt, with the same initial velocity of $5\,g = 49\ \dfrac{m}{s}$. We'll assume that our deadbolt has a measured terminal velocity of $v_\tau = -245\ \dfrac{m}{s} = -25\,g$,

so $|v_\tau| = 25\,g = \dfrac{g}{\rho} \Rightarrow \rho = .04$ (convenient). So, from our earlier work:

$v_\tau = \frac{\alpha}{\rho}$ $\quad \frac{1}{\rho} = 25$

$$v = v_\tau + (v_0 - v_\tau)\,e^{-\rho t}$$

$$y = y_0 + t\,v_\tau + \frac{(v_0 - v_\tau)\left(1 - e^{-\rho t}\right)}{\rho}$$

So,

$v_0 - v_\tau = 5g + 25g = 30g.$

$$v = -\frac{g}{\rho} + \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} = -245 + 294\,e^{-.04\,t}.$$

$$y = \underbrace{0}_{y_0 + v_\tau t} - 245\,t + \frac{294}{.04}\left(1 - e^{-.04\,t}\right).$$

When does the object reach its maximum height, what is this height, and how long does the object fall? Compare to the no-drag case with the same initial velocity, in Exercise 3.

set v(t)=0 solve for t

plug that t into y(t)

set y(t)=0 take diff ~~with~~ btwn that t & max ht t

Maple check, and then work:

```
> 
> with(DEtools) :
> g := 9.8; ρ := .04; v0 := 49;
```

$$g := 9.8$$
$$\rho := 0.04$$
$$v0 := 49 \tag{7}$$

```
> dsolve({v'(t) = -g - ρ·v(t), v(0) = v0}, v(t));
```

$$v(t) = -245 + 294\,e^{-\frac{1}{25}t} \tag{8}$$

```
> v2 := t → -245.0 + 294 e^(-1/25 t);
```

$$v2 := t \rightarrow -245.0 + 294\,e^{-\frac{1}{25}t} \tag{9}$$

```
> solve(v2(t) = 0, t);
```

$$4.5580 \tag{10}$$

compare to no drag case $t = 5$ sec. makes sense.

```
> y2 := t → y0 + ∫_0^t (-g/ρ + e^(-ρ s) (v0 + g/ρ)) ds;
```

$$y2 := t \rightarrow y0 + \int_0^t \left(-\frac{g}{\rho} + e^{-\rho s}\left(v0 + \frac{g}{\rho}\right)\right) ds \tag{11}$$

> $v2(t)$;
$\frac{294}{.04}$;
$y0 := 0$;
$y2(t)$;

$$-245.0 + 294\, e^{-\frac{1}{25} t}$$

$$7350.0$$

$$y0 := 0$$

$$\boxed{7350. - 245.\, t - 7350.\, e^{-0.040000\, t}} = 0 \quad (12)$$

solve for t, to figure out when landed.

> $solve(v2(t) = 0, t)$;
$solve(y2(t) = 0, t)$;
$y2(4.558)$;

$$\boxed{4.5580}$$

$$\boxed{9.4110, 0.}$$

$$\boxed{108.28} \quad (13)$$

dropping for

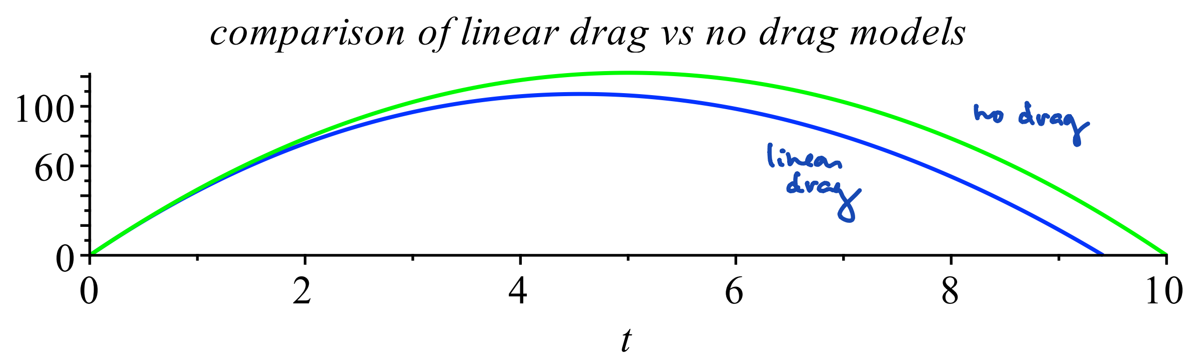
max ht

$$\begin{array}{r} 9.41 \\ -\ 4.56 \\ \hline 4.85 \end{array}$$

sec.

picture:

> $with(plots)$:
$plot1 := plot(y1(t), t = 0 ..10, color = green)$:
$plot2 := plot(y2(t), t = 0 ..9.4110, color = blue)$:
$display(\{plot1, plot2\}, title = \text{`comparison of linear drag vs no drag models`})$;



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